Kalman Filtering and Linear Quadratic Gaussian Control

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PART I - KALMAN FILTERING

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Contents

1 Introduction

We provide here some historical remarks and background on Kalman filtering and linear quadratic gaussian control.

1.1 Historical Remarks

The famous MIT mathematician Norbert Wiener developed optimal estimation theory for continuous-time problems during the 2nd World War. The solution was, however, in non-recursive form (state space methods where not yet well-known then). This made it difficult to apply the theory. Recursive solutions to optimal estimation problems were obtained by Rudolf Kalman using state space methods in the 1960's. These techniques form now the popular field of Kalman filtering. Many researchers contributed to the development of robust numerical implementations of Kalman filtering techniques. The discrete-time solutions are especially useful in applications.

Optimal linear quadratic control problems with incomplete state information were also solved in the 1960's. These developments were based on dynamic programming, developed by Richard Bellman in the 1950's, and also on Kalman filtering. The solution of the linear quadratic gaussian (LQG) control problem resulted in the famous separation theorem of LQG control. Lyapunov and Riccati matrix equations are the most important numerical objects in both Kalman filtering and LQG control.

1.2 Applications

Kalman filtering has been an important technique in various space craft, satellite, orbit tracking, and other tasks associated with various space programs, such as the famous Apollo program (astronauts landed to the moon).

Sensor fusion is an important area in which information from a number of different (types of) sensors are combined dynamically to make it possible to estimate various quantities that can not be measured directly at all or not accurately (or cost effectively) by any single sensor. Kalman filtering is a basic dynamical estimation technique in this area.

GPS (global positioning system) instruments often use Kalman filtering. More generally, Kalman filtering is an essential technique in model-based sensors.

Linear quadratic (LQ) and LQG methods were already applied in the 1960's to various missile guidance and other guidance problems. The first reported application of LQG control in process industries is by Karl Johan Åström in the mid 1960's at the IBM Nordic Laboratory. (IBM was then planning to enter the industrial process computer market.) This application involved paper machine control.

Robustness properties (against modelling uncertainty) of LQ/LQG control were studied intensively in the late 1970's and in the 1980's. These studies resulted in the LQG/LTR technique for improving robustness properties of LQG controllers. (Here LTR stands for

Loop Transfer Recovery.) These studies culminated in the development of modern robust control theory in the 1980's and in the 1990's. (H_∞ robust control is perhaps the most popular of these theories.) Kalman filtering and LQ/LQG control can be seen as the first applicable estimation and control theories based on an optimality principle.

1.3 References

The literature on Kalman filtering and LQG control is huge. We shall be very brief in our account of the literature.

The book by M.S. Grewal and A.P. Andrews (Kalman Filtering. Theory and Practice, Prentice Hall, 1993) describes the history and numerical aspects of Kalman filtering in an excellent way. The book by J.A. Farrell and M. Barth (The Global Positioning System & Inertial Navigation, McGraw-Hill, 1999) describes an important application of Kalman filtering.

The popular book by K.J. Åström and B. Wittenmark (Computer Controlled Systems. Theory and Design, Prentice-Hall) had already in its 1st edition from 1984 a good introduction to Kalman filtering and LQG control. The book by K.J. Åström (Introduction to Stochastic Control Theory, Academic Press, 1970) is a classic graduate level book on stochastic control, dealing with both continuous-time and discrete-time systems. The book by P. Dorato, C. Abdallah and V. Cerone (Linear-Quadratic Control. An Introduction, Prentice-Hall, 1995) gives an account of some of the later developments concerning LQ control, too. The many connections between modern robust control and LQ control are explained in the book by K. Zhou, J.C. Doyle and K. Glover (Robust and Optimal Control, Prentice-Hall, 1996). A classic reference on robustness of LQ and LQG control against modelling uncertainty is the book by M.G. Safonov (Stability and Robustness of Multivariable Feedback Systems, MIT Press, 1980).

2 Review of Probability Concepts

The purpose of this section is to provide a brief review of essential probability concepts.

2.1 Basic Probability Concepts

- elementary event
- event = some collection of elementary events
- sample space (event space) = set of all possible elementary events

Example 1 Play with two dice: Each die has six numbered faces (with 1,2,3,4,5, and 6 spots, respectively). After a throw of the two dices (die I and die II), each die has one

face up. An elementary event of the play is (i, j), where i and j denote the number of dots on the top face of die I and of die II, respectively. There are 36 elementary events (i, j), $i, j = 1, \ldots, 6$. The sample space Ω is thus given by

$$\Omega = \{(i, j) \mid i, j = 1, \ldots, 6\}.$$

Define event A = sum of spots is 5, i.e.

$$A = \{(1, 4), (2, 3), (3, 2), (4, 1)\}, \quad A \subset \Omega.$$

- discrete sample (event) space = sample space Ω contains only a finite or numerable number of elementary events
- continuous sample space = sample space Ω contains a denumerable number of elementary events
- set theory concepts and Venn diagrams are very useful

Example 2 The play with two dice continues! The event B = both dice have the same number of spots on the top face, i.e.

$$B = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)\}.$$

The set union $A \cup B$ means the event that either A or B takes place, and so

$$A \cup B = \{(1, 4), (2, 3), (3, 2), (4, 1), (1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)\}.$$

The set intersection $A \cap B$ means the event that both A and B take place. Clearly here $A \cap B = \emptyset$ = the empty set. That is, A and B are disjoint events.

The event G = both dice have 5 spots on the top face. Then $A \cap G = \emptyset$, $B \cup G = B$, $B \cap G = G$, and $A \cup B \cup G = A \cup B$. Note that $G \subset B$.

2.2 Kolmogorov Axioms

A probability measure $P(\cdot)$ is a function, which associates to each event A,B, etc in an event space Ω real numbers $P(A)$, $P(B)$, etc, called the probability of A, B, etc, so that the following axioms hold

$$\begin{aligned}
&\text{Axiom 1} \;:\; 0 \le P(\cdot) \le 1 \\
&\text{Axiom 2} \;:\; P(\Omega) = 1 \\
&\text{Axiom 3} \;:\; P(A \cup B) = P(A) + P(B) \quad \text{if } A \text{ and } B \text{ are disjoint}
\end{aligned}$$

or more generally

Axiom 3' : If A,B,C, ... is a numerable sequence of pairwise disjoint events, then

$$P(A \cup B \cup C \cup \ldots) = P(A) + P(B) + P(C) + \ldots.$$

$(\Omega, P(\cdot))$ is called a *probability space*. One also needs to specify the events in Ω for which $P(\cdot)$ is defined. Call this set of events $\mathcal{F}$. For discrete Ω one can simply take $\mathcal{F}$ to be the set of all subsets of Ω. For continuous Ω it is usually not convenient (or possible) to define $P(\cdot)$ for all subsets of Ω. $\mathcal{F}$ is then usually taken to be some so-called $\sigma-$algebra, i.e.

$$A \in \mathcal{F} \Rightarrow A^c = \text{complement of } A \text{ in } \Omega \in \mathcal{F}$$
$$\{A_j \in \mathcal{F}\}_{j=1}^{\infty} \Rightarrow \cup_1^{\infty} A_j \in \mathcal{F}.$$

Example 3 The play with two dice continues!

- assume that both dice are well-made (symmetrical)
- each elementary event is taken to be equally probable
- use axiom 3 repeatedly

We compute

$$P(A) = \frac{4}{36} = \frac{1}{9}, \quad P(B) = \frac{6}{36} = \frac{1}{6}$$

Furthermore

$$P(A \cup B) = \frac{10}{36} = \frac{5}{18}, \quad P(A) + P(B) = \frac{4}{36} + \frac{6}{36} = \frac{10}{36} = \frac{5}{18},$$

in agreement with axiom 3.

2.3 Conditional Probability

Define the conditional probability, $P(B \mid A)$, of event B when event A has taken place as

$$P(B \mid A) \equiv \frac{P(A \cap B)}{P(A)}, \quad P(A) > 0.$$

Example 4 The play with two dice continues! The event C = sum of spots is five (5) and both dice show a prime number. Thus $C = \{(2,3),(3,2)\}$. Then

$$P(B \mid A) = \frac{0}{P(A)} = 0, \quad B \text{ is impossible if } A \text{ has taken place}$$

$$P(C \mid A) = \frac{P(A \cap C)}{P(A)} = \frac{2/36}{4/36} = \frac{1}{2}.$$

The *law of total probability* can be stated as follows: Let $\{H_i\}_{i=1}^n$ be pairwise disjoint events, such that

$$\cup_{i=1}^n H_i = \Omega, \quad P(H_i) > 0,\ i = 1, \ldots, n.$$

Then for any event A

$$P(A) = \sum_{i=1}^{n} P(H_i)P(A \mid H_i).$$

We verify this relationship by computing:

$$P(A) = P(A \cap \Omega) = P(A \cap \cup_{i=1}^n H_i) = P(\cup_{i=1}^n (A \cap H_i)) = \sum_{i=1}^n P(A \cap H_i) = \sum_{i=1}^n P(H_i)P(A \mid H_i),$$

where the last equality follows from the definition of conditional probability.

The notion of *independence of events* is introduced as follows. If $P(B \mid A) = P(B)$ then A and B are said to be independent. We see that then $P(A \mid B)P(B) = P(A \cap B) = P(A)P(B)$ and so $P(A \mid B) = P(A)$ (assuming $P(A) > 0$ and $P(B) > 0$), i.e. independence is an equivalency relationship and furthermore then

$$P(A \cap B) = P(A)P(B).$$

This relationship is often taken as the definition of independence of the events A and B.

Example 5 The play with two dice continues! The event $D =$ die I shows 3 spots and the event $E =$ die II shows 5 spots. In terms of elementary events

$$D = \{(3,1), (3,2), \ldots, (3,6)\}, \quad E = \{(1,5), (2,5), \ldots, (6,5)\}.$$

Then

$$P(D \cap E) = \frac{1}{36} = P(D)P(E) = \frac{6}{36}\frac{6}{36} = \frac{1}{36}.$$

Note that $D \cap E = \{(3,5)\}$.

2.4 Stochastic Variables

A stochastic (random) variable (s.v.) is a function defined on a sample space,

$$X : \Omega \to \mathbb{R} \quad (\text{or } X : \Omega \to \mathbb{C}).$$

This is actually the definition of a one-dimensional s.v. (Here $\mathbb{R}$ and $\mathbb{C}$ denote the set of real numbers and the set of complex numbers, respectively.)

Definition 2.1 *The function $F_X : \mathbb{R} \to \mathbb{R}$ given by $F_X(x) = P(X \leq x)$ is called the distribution function of the s.v. X.*

Here the event $X \leq x$ means the collection of all elementary events e for which $X(e) \leq x$.

Example 6 The play with two dice continues! Let X give the sum of spots in the two dice. Then $F_X(x) = 0$ for $x < 2$, $F_X(x) = 1/36$ for $2 \leq x < 3$, $F_X(x) = 3/36$ for $3 \leq x < 4$, and so on. $F_X(x)$ has a jump at $x = 2, 3, \ldots, 12$ and $F_X(x) = 1$ for $x \geq 12$. X is a discrete s.v.

A stochastic variable (s.v.) X is *continuous* if

(a) $F_X(x)$ is continuous for all x
(b) $F_X(x)$ is differentiable with a continuous derivative except possibly at a finite number of x values

(Several different definitions appear in the literature.)

Definition 2.2 *The derivative*

$$f_X(x) = \frac{dF_X}{dx}$$

is called the (probability) density function of the stochastic variable X.

Note that for a continuous s.v. X, it holds that

$$P(a \leq X \leq b) = \int_a^b f_X(x)\, dx \ (= F_X(b) - F_x(a)).$$

Clearly $f_X(x \geq 0$ as $F_X(x)$ is a monotonically increasing function.

2.5 Multidimensional Stochastic Variables

A two dimensional stochastic variable (s.v.) is a function (X, Y) defined on an event (sample) space,

$$(X, Y) : \Omega \to \mathbb{R}^2.$$

The probability distribution function of the s.v. (X, Y) is defined as

$$F_{X,Y}(x, y) = P(X \leq x, Y \leq y),$$

where x and y are real numbers. Here the event $X \leq x$, $Y \leq y$ means the collection of all elementary events e for which $X(e) \leq x$ and $Y(e) \leq y$. Note that $F_{X,Y} : \mathbb{R}^2 \to \mathbb{R}$. Obviously, it holds that

$$0 \leq F_{X,Y}(x, y) \leq 1 \quad \text{for any } x,y.$$

The (probability) density function of the s.v. (X, Y) is defined by

$$P(a_1 \leq X \leq b_1, a_2 \leq Y \leq b_2) = \int_{a_1}^{b_1} \int_{a_2}^{b_2} f_{X,Y}(x, y)\, dx\, dy.$$

Remark 2.1 The stochastic variable (s.v.) (X, Y) is said to be *continuous* if $F_{X,Y}(x, y)$ is continuous for all x and y and if

$$\frac{\partial^2 F_{X,Y}}{\partial x \partial y} = f_{X,Y}(x, y)$$

exists and is continuous except possibly at a finite number of points $(x, y) \in \mathbb{R}^2$. Note that $f_{X,Y}(x, y) \geq 0$.

The relationship between $f_X(\cdot)$ and $f_{X,Y}(\cdot, \cdot)$ is as follows

$$P(a \leq X \leq b) = \int_a^b f_X(x)\, dx =$$
$$P(a \leq X \leq b, -\infty < Y \infty) = \int_a^b [\int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dy] dx,$$

and so

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dy.$$

Note that f_X is often called the marginal density function of X.

The *conditional distribution* of X given $Y \in I$, where I is an interval, is

$$F_{X|Y}(x \mid I) = \frac{\int_{-\infty}^{x} \left[\int_I f_{X,Y}(\xi, \eta) d\eta\right] d\xi}{\int_{-\infty}^{\infty} \left[\int_I f_{X,Y}(\xi, \eta) d\eta\right] d\xi},$$

where it is assumed that the denominator is greater than zero. The *conditional density function* of X given $Y \in I$ is

$$f_{X|Y}(x \mid I) = \frac{\int_I f_{X,Y}(x, \eta) d\eta}{\int_{-\infty}^{\infty} \left[\int_I f_{X,Y}(\xi, \eta) d\eta\right] d\xi}$$

Denote $I_\delta = I = [y - \delta/2, y + \delta/2]$, where $\delta > 0$. Let $f_{X,Y}(\xi, \eta)$ be continuous at (x, y). Define

$$f_{X|Y}(x \mid y) \equiv \lim_{\delta \to 0} f_{X|Y}(x \mid I_\delta) = \frac{f_{X,Y}(x, y)}{f_Y(y)},$$

where it is assumed that $f_Y(y) > 0$. (Here $f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dx$ is the marginal density function of Y.) The function $f_{X|Y}(x \mid y)$ is the conditional (probability) density function of X given $Y = y$.

The stochastic variables X and Y are said to be independent if

$$f_{X|Y}(x \mid y) = f_X(x), \quad \text{i.e. if}$$

$$f_{X,Y}(x, y) = f_X(x) f_Y(y)$$

holds for any x and y.

The expectation, mean value, of a stochastic variable X is defined as

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx$$

(for a discrete stochastic variable $E[X] = \sum_x xP(X = x)$). Note that $E[X]$ need not be well-defined, i.e. the integral in the definition might not exist. If $\int_{-\infty}^{\infty} |x f_X(x)|\, dx < \infty$ then $E[X]$ exists (as a real number).

Example 7 The Cauchy distribution has the density function

$$f_X(x) = \frac{1}{\pi(1 + x^2)}.$$

Then

$$E[X] = \lim_{a \to -\infty, b \to \infty} \int_a^b x \frac{1}{\pi(1 + x^2)}\, dx,$$

but clearly the double limit does not exist, i.e. $E[X]$ does not exist as a real number.

Let us now consider the expected value of functions of a stochastic variable (s.v.). Let $Z = g(X)$, where g is a function of the s.v. X. The expected value of Z is then

$$E[Z] = \int_{-\infty}^{\infty} z f_z(z)\, dz = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx, \quad \text{i.e.}$$

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx,$$

when the above integral exists.

The conditional mean of $Z = g(X)$ given $Y = y$ (here X and Y are dependent s.v.s) is

$$E[Z \mid Y = y] = \int_{-\infty}^{\infty} f_{Z|Y}(z \mid y)\, dz = \int_{-\infty}^{\infty} g(x) f_{X|Y}(x \mid y)\, dx, \quad \text{i.e.}$$

$$E[g(X) \mid Y = y] = \int_{-\infty}^{\infty} g(x) f_{X|Y}(x \mid y)\, dx.$$

An important relationship for expectations is obtained as follows.

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx = \int_{-\infty}^{\infty} g(x) [\int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dy] dx =$$
$$\int_{-\infty}^{\infty} [\int_{-\infty}^{\infty} g(x) f_{X|Y}(x \mid y) dx] f_Y(y)\, dy = \int_{-\infty}^{\infty} E[g(X) \mid Y = y] f_Y(y)\, dy = E_Y[E[g(X) \mid Y]],$$

where $E_Y[\cdot]$ denotes expectation over the s.v. Y. Here we have assumed that the order of integration can be changed (this is true if the associated double integral exists). The above relationship expresses *the law of total probability for expectations*, that is

$$E[g(X)] = E_Y[E[g(X) \mid Y]]. \tag{1}$$

Check: If X and Y are independent s.v.s then $E[g(X) \mid Y = y] = E[g(X)]$ and the right hand side of (1) reduces to $E[g(X)]$.

The variance of a stochastic variable is defined as

$$\text{var}(X) = E[(X - E[X])^2] = \int_{-\infty}^{\infty} (x - E[X])^2 f_X(x)\, dx.$$

Similarly, the covariance of the s.v.s X and Y is defined as

$$\text{cov}(X, Y) = E[(X - E[X])(Y - E(Y))] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x - E[X])(y - E[Y]) f_{X,Y}(x, y)\, dxdy$$

Remark 2.2 For $Z = g(X, Y)$ we have that

$$E[Z] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x, y) f_{X,Y}(x, y)\, dxdy$$

when the double integral exists.

We have so far only considered one-dimensional and two-dimensional stochastic variables. Vector-valued stochastic variables are introduced in the following manner. Let X_i, $i = 1, 2, \ldots, n$, be mutually dependent (or possibly independent) stochastic (random) variables. $X = [X_1, X_2, \ldots, X_n]^T$ is then said to be an $n-$dimensional stochastic variable. (Here the superscript ^{T} denotes vector transpose.)

Define the expected value of X as

$$E[X] = \begin{pmatrix} E[X_1] \\ \vdots \\ E[X_n] \end{pmatrix} \in \mathbb{R}^n,$$

where $\mathbb{R}^n$ denotes the (Euclidean) space of all $n-$tuples of real numbers. The covariance matrix of X is defined as

$$[\text{cov}(X)]_{ij} = E[(X_i - E[X_i])(X_j - E[X_j])]$$

or in compact matrix notation

$$\text{cov}(X) = E[(X - E[X])(X - E[X])^T]$$

Note that $\text{cov}(X)$ is a symmetric positive semidefinite matrix, i.e.

$$[\text{cov}(X)]^T = \text{cov}(X) \quad \text{and} \quad v^T \text{cov}(X) v \geq 0$$

for all $v \in \mathbb{R}^n$.

2.6 Stochastic Processes

Many applications of stochastic variables (s.v.s) involve families of s.v.s called stochastic processes.

A *stochastic process* is a family of stochastic variables X_t, where t is a parameter running over a suitable index set T (sometimes we write $X(t)$ instead of X_t). Here T could be e.g. the set of all integers $\mathbb{Z}$, the set of natural numbers $\mathbb{N}$, or the interval $[0, \infty)$. A *realization* or sample function of a stochastic process $\{X_t, t \in T\}$ is an assignment, to each $t \in T$, of a possible value of X_t. For example T could correspond to discrete units of time $T = \mathbb{N} = \{0, 1, \ldots\}$ and $\{X_t\}$ could then represent the outcomes at successive trials (throwing dice etc.).

$(X(t_1), X(t_2), \ldots, X(t_m))^T$, where t_i, $i = 1, 2, \ldots, m$, are m distinct index values, is an $m-$dimensional stochastic variable. The function

$$F_X(x_1, \ldots, x_m; t_1 \ldots, t_m) = P(X(t_1) \leq x_1, \ldots, X(t_m) \leq x_m)$$

denotes the probability distribution function of this $m-$dimensional s.v. ($x_i \in \mathbb{R}$, $i = 1, \ldots, m$, are real numbers). Similarly, the function $f_X(x_1, \ldots, x_m; t_1, \ldots, t_m)$ defined by

$$F_X(x_1, \ldots, x_m; t_1, \ldots, t_m) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_m} f_X(\xi_1, \ldots, \xi_m; t_1, \ldots, t_m)\, d\xi_1 \cdots d\xi_m,$$

is called the probability density function of the s.v. $(X(t_1), X(t_2), \ldots, X(t_m))^T$.

The *mean value function* of the stochastic process $\{X(t)\}$ is defined by

$$m_X(t) = E[X(t)] = \int_{-\infty}^{\infty} x f_X(x; t)\, dx.$$

(Here $X(t)$ is a one-dimensional stochastic variable for any $t \in T$.) The *covariance function* of the stochastic process $X(t)$ (we simplify the notation $\{X(t)\}$, $(X(t))$ and $(X(t))_{t \in T}$ often by writing $X(t)$ for the stochastic process in question) is defined by

$$r_{XX}(t) = \mathrm{cov}[X(s), X(t)] = E[(X(s) - m_X(s))(X(t) - m_X(t))] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x_1 - m_X(s))(x_2 - m_X(t)) f_X(x_1, x_2; s, t)\, dx_1 dx_2.$$

We shall often need to consider the case when $\{X(t), t \in T\}$ is a vector stochastic process, i.e. when $X(t)$ is a function from some event space Ω into the $n-$dimensional Euclidean space $\mathbb{R}^n$, i.e. when

$$X(t) = \begin{pmatrix} X_1(t) \\ \vdots \\ X_n(t) \end{pmatrix}$$

Then

$$m_X(t) = E[X(t)] = \begin{pmatrix} E[X_1(t)] \\ \vdots \\ E[X_n(t)] \end{pmatrix}$$

is a function from T into $\mathbb{R}^n$. Similarly the covariance function $X(t)$ is then

$$r_{XX}(t) = \text{cov}[X(s), X(t)] = E[(X(s) - m_X(s))(X(t) - m_X(t))^T] = E[\{(X_i(s) - (m_X)_i(s))(X_j(t) - (m_X)_j(t))\}_{n\times n}].$$

(The notation $\{a_{ij}\}_{n\times n}$ means an $n \times n$ matrix with general element a_{ij}.)

A stochastic process $X(t)$ is called *stationary* if the probability distribution function of $X(t_1)$,. . .,$X(t_m)$ is identical to the distribution function of $X(t_1+\tau)$,. . .,$X(t_m+\tau)$ for any $m \geq 1$, for any distinct t_1,. . ., t_m in T and for any τ such that $t_1 + \tau$, . . ., $t_m + \tau$ are in T, too. Note that we denote compactly

$$F_X(x_1, \ldots, x_m; t_1, \ldots, t_m) = P(X(t_1) \leq x_1, \ldots, X(t_m) \leq x_m)$$

as the probability distribution function of $X(t_1)$,. . .,$X(t_m)$, where $x_i \in \mathbb{R}^n$, $i = 1, \ldots, m$, and the inequalities are vector inequalities! (So here the vector inequality $X(t_1) \leq x_1$ means that $X_j(t_1) \leq (x_1)_j$ for all $j = 1, \ldots, n$ and $(x_1)_j \in \mathbb{R}$ denotes the $j-$th component of the vector $x_1 \in \mathbb{R}^n$.) The condition for stationarity is then

$$F_X(x_1, \ldots, x_m; t_1 + \tau, \ldots, t_m + \tau) = F_X(x_1, \ldots, x_m; t_1, \ldots, t_m)$$

for all possible (admissible) choices of m, x_1, . . ., x_m, t_1, . . ., t_m and τ. Here usually the index variable t ($t \in T$) is a time variable!

Note that we could order the distribution functions $F_{X_i}(x; t)$, $i = 1, \ldots, n$, in a vector (here $X_i(t)$ is a one-dimensional s.v. and so x is a scalar, i.e. $x \in \mathbb{R}$)

$$\underline{F}_X(x; t) \equiv \begin{pmatrix} F_{X_1}(x; t) \\ \vdots \\ F_{X_n}(x; t) \end{pmatrix}$$

and similarly the density functions $f_{X_i}(x; t)$,$i = 1, \ldots, n$, in a vector $\underline{f}_X(x; t)$, and write compactly

$$E[X(t)] = \int_{-\infty}^{\infty} x \underline{f}_X(x; t)\, dx = \int_{-\infty}^{\infty} x\, d\underline{F}_X(x; t) \in \mathbb{R}^n.$$

(Here we have expressed the mean value of the multi-dimensional s.v. $X(t)$ using the marginal distribution functions $F_{X_i}(x, t)$ for the one-dimensional s.v.s. $X_i(t)$, $i = 1, \ldots, n$.)

Note that if

$$m_X(t) = \text{ constant for all } t$$

and $r_{XX}(s, t)$ depends only on the difference $s - t$, then the stochastic process $(X(t))$ is called *weakly stationary*. A stationary s.p. (stochastic process) is also weakly stationary, but the reverse need not hold.

A s.p. $(X(t))$ with covariance function

$$r_{XX}(t, t+\tau) = \delta(\tau)\Sigma,$$

where the Kronecker delta symbol $\delta(\tau) = 1$ if $\tau = 0$ and $\delta(\tau) = 0$ for $\tau \neq 0$, is called (discrete-time) *white noise* when T is a discrete-time index set (e.g. $T = \mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$). Here Σ denotes a symmetric positive semidefinite matrix of dimensions $n \times n$ ($X(t)$ has n components).

3 Stochastic State Space Models

In this section we discuss state space models or state space systems in a stochastic context. Such models are central in Kalman filtering and LQG control.

3.1 Discrete State Space Systems

We consider linear stochastic discrete-time systems described by the (state space) equations

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) + w(t) \quad (2) \\ y(t) &= Cx(t) + v(t), \quad (3) \end{aligned}$$

where $x(\cdot)$ is the state vector (of dimension n), $u(\cdot)$ is the input vector (of dimension p), $y(\cdot)$ is the output vector, t denotes discrete time (normalized by dividing with the sampling time) $t \in \mathbb{N} = \{0, 1, 2, \ldots\}$ (or $t \in \mathbb{Z}$). Furthermore A, B and C are matrices of appropriate dimensions, and $\{w(t)\}$ and $\{v(t)\}$ are white noise processes with zero mean values and with the covariance matrices

$$\begin{aligned} E[w(t)w(s)^T] &= \begin{cases} R_1, & \text{if } t = s \\ 0, & \text{if } t \neq s \end{cases} \\ E[v(t)v(s)^T] &= \begin{cases} R_2, & \text{if } t = s \\ 0, & \text{if } t \neq s \end{cases} \\ E[w(t)v(s)^T] &= \begin{cases} R_{12}, & \text{if } t = s \\ 0, & \text{if } t \neq s \end{cases} \end{aligned}$$

where R_1, R_2 and R_{12} are given matrices (so-called covariance matrices). Furthermore it is assumed that

$$E[w(t)x(t)^T] = 0 \quad \text{and} \quad E[v(t)x(t)^T] = 0.$$

Often (in books etc.) it is also assumed that

$$R_{12} = 0$$

as this simplifies many derivations and expressions.

We should note that it is often natural to interpret (3) as the measurement equation ($y(\cdot)$ denotes then measured signals). In (2)-(3) the terms $w(t)$ and $v(t)$ are called the the process noise and the measurement noise, respectively.

Remark 3.1 In applications the state space model (2)-(3) is often a sampled model of a continuous-time system.

Let us consider the computation of the mean value (function) of the state $x(t)$. Consider the case that $u(t) = 0$ for all t equal or greater than some initial time t_0. Denote $m_x(t) = E[x(t)]$. Then by (2)

$$m_x(t+1) = Am_x(t) + E[w(t)] = Am_x(t), \quad t \geq t_0 \tag{4}$$

as $w(t)$ has zero mean value by assumption. Note that this difference equation has the initial value $m_x(0) = E[x(0)]$ assuming $t_0 = 0$ to be the initial time.

Using (4) repeatedly we get that

$$m_x(t) = A^t m_x(0), \quad t \geq 0.$$

Hence $m_x(t) \to 0$ as $t \to \infty$ if A is a stable matrix, i.e. if all the eigenvalues of A are located strictly inside the unit circle (that is, if all the solutions λ of $\det(A - \lambda I) = 0$ satisfy $|\lambda| < 1$).

The covariance matrix of the state $x(t)$ is defined as

$$P_x(t) = E[(x(t) - m_x(t))(x(t) - m_x(t))^T].$$

Let us evaluate the state covariance matrix when $u(t) = 0$ for all $t \geq 0$. Then (2) and (4) give that

$$\begin{aligned} P_x(t+1) = E[(Ax(t) + w(t) - m_x(t+1))(Ax(t) + w(t) - m_x(t+1))^T] = \\ E[(A(x(t) - m_x(t)) + w(t))(A(x(t) - m_x(t)) + w(t))^T] = \\ E[A(x(t) - m_x(t))(A(x(t) - m_x(t))^T] + R_1 = AP_x(t)A^T + R_1 \end{aligned} \tag{5}$$

as $w(t)$ has zero mean and as $E[x(t)w(t)^T] = (E[w(t)x(t)^T])^T = 0$ by assumption.

Let $P_x(0)$ denote the initial value of $P_x(t)$. What happens when $t \to \infty$? By repeated use of (5) we get that

$$\begin{aligned} P_x(t) &= A(AP_x(t-1)A^T + R_1)A^T + R_1 \\ &= A^2 P_x(t-1)(A^T)^2 + AR_1A^T + R_1 \\ &= A^2(AP_x(t-2)A^T + R_1)(A^T)^2 + AR_1A^T + R_1 \\ &= A^3 P_x(t-2)(A^T)^3 + A^2R_1(A^T)^2 + AR_1A^T + R_1 \\ &= \ldots \qquad (6) \\ &= A^{t+1}P_x(0)(A^T)^{t+1} + \sum_{s=0}^{t} A^s R_1 (A^T)^s \qquad (7) \end{aligned}$$

If A has all its eigenvalues strictly inside the unit circle then $A^t \to 0$ as $t \to \infty$ (here 0 denotes the zero matrix of size $n \times n$) in such a way that the limit

$$\lim_{t\to\infty} \sum_{s=0}^{t} A^s R_1 (A^T)^s$$

exists. Hence $P_x \equiv \lim_{t\to\infty} P_x(t)$ exists in this case and P_x is then given by

$$P_x = \sum_{s\geq 0} A^s R_1 (A^T)^s \tag{8}$$

Note that by (8)

$$AP_xA^T = P_x - R_1$$

or equivalently

$$P_x = AP_xA^T + R_1 \tag{9}$$

This equation is a linear matrix equation in (the unknown symmetric, positive semidefinite matrix) P_x. Due to the great importance of this equation, it has its own name – the algebraic discrete matrix Lyapunov equation. The difference equation (5) is often called the discrete Lyapunov difference equation.

Example 8 Let $\dim x = n = 1$. Then (7) reduces to (denoting $a = A$ and $r_1 = R_1$ as here A and R_1 are real numbers, i.e. scalars)

$$p_x(t) = a^{2t+2} p_x(0) + \sum_{s=0}^{t} a^{2s} r_1,$$

where we use the notation $p_x = P_x$ as P_x is here a scalar. For $|a| < 1$ the sum in this equation is a convergent geometric series and so

$$\sum_{s\geq 0} a^{2s} r_1 = \frac{r_1}{1-a^2}.$$

The limiting value of $p_x(t)$ is thus given by

$$p_x = \lim_{t\to\infty} p_x(t) = \frac{r_1}{1-a^2}$$

as $\lim_{t\to\infty} a^{2t+2} p_x(0) = 0$ for any $p_x(0)$ when $|a| < 1$.

Remark 3.2 Consider the deterministic state space system

$$x(t+1) = Ax(t) \tag{10}$$

with given initial state $x(0) = x_0$. (Here A is a stable matrix.) Let us look at the quantity

$$\overline{P} = \sum_{s\geq 0} x(s)x(s)^T.$$

By (10)

$$A\overline{P}A^T = \sum_{s\geq 0} Ax(s)(Ax(s))^T = \sum_{s\geq 0} x(s+1)x(s+1)^T = \sum_{k\geq 1} x(k)x(k)^T = \overline{P} - x(0)x(0)^T$$

and so

$$\overline{P} = A\overline{P}A^T + R, \tag{11}$$

where $R = x(0)x(0)^T$. The similarity of (9) and (10) implies that the same numerical algorithms and software can be used to solve both stochastic problems and deterministic initial value problems.

3.2 Numerical Solution of the Lyapunov Equation

The algebraic Lyapunov matrix equation that we consider here is of the form (9) or (11), i.e.

$$P = APA^T + R, \tag{12}$$

where A is a given STABLE square matrix of size $n \times n$, P is an unknown symmetric positive semidefinite matrix and R is a given symmetric positive semidefinite matrix, meaning that $R = R^T$ and $x^T Rx \geq 0$ for any $x \in \mathbb{R}$.

Methods to solve (12):

1) Equation (12) is a *linear system of equations* in the unknown elements P_{ij}, $j \leq i$, $i = 1, 2, \ldots, n$, of P (i.e. there are $(n(n+1)/2$ unknown variables). It is, however, not practical to solve (12) as a standard linear equation system problem (say e.g. using Gauss elimination) especially when n is large ($n(n+1)/2$ grows very fast, e.g. for a system with 20 state variables $n = 20$, and so $n(n+1)/2 = 210$, and with $n = 200$, $n(n+1)/2 = 21000$, i.e. already a VERY BIG system of linear equations).

2) *Direct iteration:* Here one iterates as

$$P_{k+1} = AP_kA^T + R_1, \quad P_0 > 0,$$

starting with an arbitrary symmetric positive definite matrix P_0 (positive definite matrix means that $x^T P_0 x > 0$ for any nonzero $x \in \mathbb{R}^n$). The convergence of P_k to a solution can be very slow, especially if A has one or more eigenvalues near the unit circle. (Obviously convergence does not take place if A is not a stable matrix.)

3) *Accelerated iteration* – a simple and fast method. Here one defines the sequence $(S_k)_{k\geq 1}$ by

$$\begin{aligned} S_1 &= R \\ S_2 &= S_1 + AS_1A^T \\ S_3 &= S_2 + A^2S_2(A^2)^T \end{aligned}$$

$$\begin{aligned} S_4 &= S_3 + A^4 S_3 (A^4)^T \\ \vdots &= \vdots \\ S_{k+1} &= S_k + A^{2^{k-1}} S_k (A^{2^{k-1}})^T, \quad k \geq 1, \end{aligned}$$

the last equation representing how the general element S_{k+1} is obtained from S_k. Clearly

$$S_{k+1} = R + ARA^T + \ldots + A^{2^{k-1}} R (A^{2^{k-1}})^T,$$

so that $S_k \to P$ when A is stable. The number of terms doubles in the above sum at each iteration step.

Note that by writing $W_k = A^{2^{k-1}}$, we see that the accelerated iteration procedure can be put in the form

$$\begin{aligned} S_1 &= R, \quad W_1 = A \\ S_2 &= S_1 + W_1 S_1 W_1^T, \quad W_2 = W_1^2 \\ S_3 &= S_2 + W_2 S_2 W_2^T, \quad W_3 = W_2^2 \\ \vdots &= \vdots \\ S_{k+1} &= S_k + W_k S_k W_k^T, \quad W_{k+1} = W_k^2. \end{aligned}$$

Due to the matrix multiplications involved in this method, it can be sometimes difficult to get very high-accuracy solutions with this method. Matrix transformations to real Schur form are used in certain software, cf MATLAB$^{\text{TM}}$.

3.3 ARMAX and State Space Models

AutoRegressive Moving Average with eXternal input (ARMAX) models are a popular class of input-output models of the form

$$\begin{aligned} y(t+1) = A_1 y(t) + \ldots + A_q y(t-q+1) + B_0 u(t-L) + \ldots + \\ B_r u(t-L-r) + e(t+1) + \ldots + C_s e(t-s+1), \end{aligned}$$

where A_i, B_j and C_k are matrices of appropriate dimensions, y is the output, u is the input and e is white noise with zero means. Furthermore, $L \geq 0$ is the input delay. ARMAX models are quite popular in various applications. Note that ARMAX models can be written as state space models in many ways. E.g. one can introduce (for $L = 1, 2 \ldots$) the state vector $x(t) = [y(t)^T, \ldots, y(t-q+1)^T, u(t-1)^T, \ldots, u(t-L)^T, e(t)^T, \ldots, e(t-s+1)^T]^T$ and rewrite the ARMAX model in state space form. Note that the so obtained state space model need not be a minimal realization of the ARMAX model (i.e. a state space model of the lowest possible state space dimension with the same input-output mapping as the ARMAX model).

Example 9 Consider the ARMAX system

$$y(t+1) = ay(t) + bu(t) + e(t+1) + ce(t),$$

where it is assumed that $a + c \neq 0$ and $|c| < 1$ and $E[e(t)^2] > 0$.

Introduce the state vector $x(t) = y(t) - e(t)$. Then

$$\begin{aligned} x(t+1) &= ax(t) + bu(t) + (a+c)e(t) \\ y(t) &= x(t) + e(t) \end{aligned}$$

is a minimal state space realization of the ARMAX system. Denote $w(t) = (a + c)e(t)$ and so $w(t)$ is the process noise term in the above state space system. Note that here

$$E[w(t)e(t)] = (a + c)E[e(t)^2] > 0$$

and so in this case the process and measurement noise terms are correlated!

Now introduce $x_1(t) = y(t)$ and $x_2(t) = e(t)$. Then the ARMAX system can be written as (i.e. it has the nonmimimal state space realization:)

$$\begin{aligned} \begin{pmatrix} x_1(t+1) \\ x_2(t+1) \end{pmatrix} &= \begin{pmatrix} a & c \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} b \\ 0 \end{pmatrix} u(t) + \begin{pmatrix} 1 \\ 1 \end{pmatrix} e(t+1) \\ y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \end{aligned}$$

Here

$$w(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e(t+1)$$

is the process noise term. Clearly

$$R_1 = E[w(t)w(t)^T] = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} E[e(t)^2]$$

and $R_2 = 0$, i.e. the covariance matrix (or here the variance) of the measurement noise term is zero as there is no measurement noise term present in this state space realization.

4 Optimal Estimation

In this section we study optimal estimation in the sense of a mean square estimation error criterion as necessary preparation for the study of optimal estimation of the state of a state space system from measured output values.

4.1 Minimum Variance Estimation

Here we start with general minimum variance estimation without assuming a dynamic context. Thus let X and Y be mutually dependent (possibly vector-valued) stochastic variables. The task is to estimate the value of X given that $Y = y$ (i.e. that the stochastic variable Y has obtained the value y). The estimate $\hat{x}$ of X given $Y = y$ is taken to be a function of the information about Y only, i.e. $\hat{x} = \hat{x}(y)$.

We shall use the popular *conditional mean square error* as a measure of the estimation error. The condition mean square estimation error is the quantity

$$E[(X-\hat{x})^T(X-\hat{x}) \mid Y=y]. \tag{13}$$

The *minimum mean square estimate* (minimum variance estimate, least squares estimate) $\hat{x}$ of X given $Y=y$ is defined by the condition

$$E[(X-\hat{x})^T(X-\hat{x}) \mid Y=y] \leq E[(X-z)^T(X-z) \mid Y=y] \tag{14}$$

which is to hold for every allowable estimate $z=z(y)$ of X given $Y=y$.

Can we characterize the best estimate $\hat{x}$ in the sense of (14) more explicitly? It turns out that the minimum mean square estimate (MMS estimate) is given uniquely as the conditional mean of X given $Y=y$, i.e.

$$\hat{x} = E[X \mid Y=y] = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} x f_{X|Y}(x \mid y)\, dx_1 \cdots dx_n, \tag{15}$$

where $n = \dim x$. Here $f_{X|Y}(x \mid y)$ denotes the (conditional) joint density function of $X_1, \ldots, X_n$ given $Y=y$ (recall that $X=[X_1, \ldots, X_n]^T$).

This is seen as follows. Let $z=z(y)$ denote any admissible estimate of X given $Y=y$. Then the conditional mean square error can be expressed as

$$\begin{gathered} E[(X-z)^T(X-z) \mid Y=y] = E[X^TX \mid Y=y] - 2z^TE[X \mid Y=y] + z^Tz = \\ (z-E[X \mid Y=y])^T(z-E[X \mid Y=y]) + E[X^TX \mid Y=y] \\ -(E[X \mid Y=y])^TE[X \mid Y=y]. \end{gathered}$$

Here $(z-E[X \mid Y=y])^T(z-E[X \mid Y=y]) \geq 0$ for any z and the terms $E[X^TX \mid Y=y]$ and $(E[X \mid Y=y])^TE[X \mid Y=y]$ are independent of z. Hence

$$E[(X-z)^T(X-z) \mid Y=y] \geq E[X^TX \mid Y=y] - (E[X \mid Y=y])^TE[X \mid Y=y]$$

and the lower bound is attained by the estimate

$$z = E[X \mid Y=y]$$

which is thus the MMS estimate of the value of X given $Y=y$.

Remark 4.1 Note that for the estimate $x=x(y)$, it holds that

$$E[(X-x(Y))^T(X-x(Y))] = E_Y[E[(X-x(Y))^T(X-x(Y)) \mid Y]] \tag{16}$$

by the law of total probability for the expectation $E[g(X,Y)]$:

$$E[g(X,Y)] = E_Y[E[g(X,Y) \mid Y]].$$

Hence $\hat{x}(y) = E[X \mid Y=y]$ minimizes also the unconditional expectation (16).

The function $\hat{x}(Y)$ is called an estimator; it is a stochastic variable!

4.2 Interlude: Properties of Normally Distributed Stochastic Variables

Here we consider properties of an important class of stochastic variables (s.v.s), namely normally distributed s.v.s.

An $n_X -$ dimensional normally distributed vector (s.v.) X with mean value m_X and the covariance matrix R_X has the probability density function

$$f_x(x) = \frac{1}{(2\pi)^{n_X/2}(\det R_X)^{1/2}} \exp[-\frac{1}{2}(x - m_X)^T R_x^{-1}(x - m_X)] \tag{17}$$

Here it is assumed that the matrix inverse R_X^{-1} exists $\Leftrightarrow \det R_X \neq 0$: (if R_X^{-1} does not exist, then x can be written as $x = T\bar{x}$ with $\dim \bar{x} < \dim x$, where $\bar{x}$ has a nonsingular covariance matrix.)

Properties: Let X and Y be jointly normally distributed vectors. Denote

$$Z = \begin{pmatrix} X \\ Y \end{pmatrix}.$$

Now Z is normally distributed with mean value denoted as

$$m = \begin{pmatrix} m_X \\ m_Y \end{pmatrix}$$

and the (symmetric) covariance matrix denoted as

$$R = \begin{pmatrix} R_X & R_{XY} \\ R_{YX} & R_Y \end{pmatrix}. \tag{18}$$

Here $R_{XY}^T [= (R_{XY})^T] = R_{YX}$ as $R^T = R$.

Then:

(a) The stochastic variables X and Y are independent if and only if they are uncorrelated.

(b) The conditional distribution of X given $Y = y$ is also normal with mean value

$$m_{X|Y} = E[X \mid Y = y] = m_X + R_{XY}R_Y^{-1}(y - m_Y) \tag{19}$$

and the covariance matrix

$$R_{X|Y} = E[(X - m_{X|Y})(X - m_{X|Y})^T \mid Y = y] = R_X - R_{XY}R_Y^{-1}R_{YX} \tag{20}$$

(c) The stochastic variables Y and $X - m_{X|Y}$ are independent.

The above properties can be derived as follows.

(a) As independence implies the uncorrelated property, we only need to establish that if X and Y are uncorrelated (normally distributed s.v.s) then they are independent. Thus consider the conditional probability distribution function

$$f_{X|Y}(x \mid y) = \frac{f_{X,Y}(x, y)}{f_Y(y)} \tag{21}$$

where (cf. (17))

$$f_Y(y) = \frac{1}{(2\pi)^{n_Y/2}(\det R_Y)^{1/2}} \exp[-\frac{1}{2}(y-m_Y)^T R_y^{-1}(y-m_Y)] \tag{22}$$

and n_Y is the dimension of Y. We need to show that $f_{X|Y}(x \mid y) = f_X(x)$ (or which is the same thing that $f_{X,Y}(x,y) = f_X(x)f_Y(y)$) if X and Y are uncorrelated.

We proceed to evaluate $f_{X,Y}(x,y)$ (recall that $Z = [X^T, Y^T]^T$):

$$f_{X,Y}(x,y) = f_Z(z) = \frac{1}{(2\pi)^{n_Z/2}(\det R)^{1/2}} \exp[-\frac{1}{2}(z-m)^T R^{-1}(z-m)], \tag{23}$$

where $n_Z = n_X + n_Y$ is the dimension of Z. We need to express R^{-1} and $\det R$ in terms of R_X, R_{XY}, R_{YX} and R_Y in R in (18). Introduce the auxiliary matrices

$$A = \begin{pmatrix} I & -R_{XY}R_Y^{-1} \\ 0 & I \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} I & 0 \\ -R_Y^{-1}R_{YX} & I \end{pmatrix},$$

where I denotes an identity matrix of appropriate size (so I is a generic notation for an identity matrix of arbitrary size). Then

$$\begin{aligned} ARB &= \begin{pmatrix} I & -R_{XY}R_Y^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} R_X & R_{XY} \\ R_{YX} & R_Y \end{pmatrix} \begin{pmatrix} I & 0 \\ -R_Y^{-1}R_{YX} & I \end{pmatrix} \\ &= \begin{pmatrix} R_X - R_{XY}R_Y^{-1}R_{YX} & 0 \\ R_{YX} & R_Y \end{pmatrix} \begin{pmatrix} I & 0 \\ -R_Y^{-1}R_{YX} & I \end{pmatrix} \\ &= \begin{pmatrix} R_X - R_{XY}R_Y^{-1}R_{YX} & 0 \\ 0 & R_Y \end{pmatrix}. \end{aligned}$$

By the rules for computing determinants of products of matrices, we get that

$$\det(ARB) = \det A \times \det R \times \det B = 1 \times \det R \times 1 = \det R.$$

Hence

$$\det R = \det \begin{pmatrix} R_X - R_{XY}R_Y^{-1}R_{YX} & 0 \\ 0 & R_Y \end{pmatrix} = \det(R_X - R_{XY}R_Y^{-1}R_{YX}) \det R_Y. \tag{24}$$

Now

$$\begin{pmatrix} I & R_{XY}R_Y^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} I & -R_{XY}R_Y^{-1} \\ 0 & I \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} = I_{n_Z \times n_Z},$$

where we have stressed that the last identity matrix has n_Z rows (and n_Z columns). Thus the inverse of A is given by

$$A^{-1} = \begin{pmatrix} I & R_{XY}R_Y^{-1} \\ 0 & I \end{pmatrix}.$$

Furthermore

$$\begin{pmatrix} I & 0 \\ R_Y^{-1}R_{YX} & I \end{pmatrix} \begin{pmatrix} I & 0 \\ -R_Y^{-1}R_{YX} & I \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix},$$

and so the inverse of B is given by

$$B^{-1} = \begin{pmatrix} I & 0 \\ R_Y^{-1}R_{YX} & I \end{pmatrix}.$$

Now we are ready to determine an expression for the inverse of R. Denote $C = (ARB)^{-1}$. Observe that

$$\begin{aligned} C = (ARB)^{-1} &= \begin{pmatrix} R_X - R_{XY}R_Y^{-1}R_{YX} & 0 \\ 0 & R_Y \end{pmatrix}^{-1} \\ &= \begin{pmatrix} (R_X - R_{XY}R_Y^{-1}R_{YX})^{-1} & 0 \\ 0 & R_Y^{-1} \end{pmatrix} \end{aligned}$$

and as $(ARB)^{-1} = B^{-1}R^{-1}A^{-1}$, we get that

$$R^{-1} = BCA. \tag{25}$$

Thus

$$(z-m)^T R^{-1}(z-m) = ((x-m_X)^T, (y-m_Y)^T)R^{-1}\begin{pmatrix} x-m_X \\ y-m_Y \end{pmatrix} =$$
$$((x-m_X)^T, (y-m_Y)^T)\begin{pmatrix} I & 0 \\ -R_Y^{-1}R_{YX} & I \end{pmatrix} C \begin{pmatrix} I & -R_{XY}R_Y^{-1} \\ 0 & I \end{pmatrix}\begin{pmatrix} x-m_X \\ y-m_Y \end{pmatrix} =$$
$$[x^T - m_X^T - (y-m_Y)^T R_Y^{-1}R_{YX}, y^T - m_Y^T]C\begin{pmatrix} x-m_X-R_{XY}R_Y^{-1}(y-m_Y) \\ y-m_Y \end{pmatrix} =$$
$$\begin{pmatrix} x-m_X-R_{XY}R_Y^{-1}(y-m_Y) \\ y-m_Y \end{pmatrix}^T \times$$
$$\begin{pmatrix} (R_X - R_{XY}R_Y^{-1}R_{YX})^{-1}(x-m_X-R_{XY}R_Y^{-1}(y-m_Y)) \\ R_Y^{-1}(y-m_Y) \end{pmatrix} =$$
$$(x-m_X-R_{XY}R_Y^{-1}(y-m_Y))^T(R_X - R_{XY}R_Y^{-1}R_{YX})^{-1}(x-m_X-R_{XY}R_Y^{-1}(y-m_Y))$$
$$+(y-m_Y)^T R_Y^{-1}(y-m_Y)$$

Combining the above result with (21)-(25), we see that the conditional density function $f_{X|Y}(x \mid y)$ is given by

$$\begin{aligned} f_{X|Y}(x \mid y) = K\exp[-\tfrac{1}{2}(x-m_X-R_{XY}R_Y^{-1}(y-m_Y))^T(R_X - R_{XY}R_Y^{-1}R_{YX})^{-1} \times \\ (x-m_X-R_{XY}R_Y^{-1}(y-m_Y))], \end{aligned} \tag{26}$$

where

$$K = \frac{1}{(2\pi)^{n_X/2}[\det(R_X - R_{XY}R_Y^{-1}R_{YX})]^{1/2}}.$$

Note that if X and Y are uncorrelated then $R_{XY} = 0$ and $R_{YX} = 0$ so then by (26) $f_{X|Y}(x \mid y) = f_X(x)$. We have therefore established that property (a) holds.

Furthermore, we see that (b) holds by inspection from (26)! To establish that (c) holds, we compute the the covariance matrix $E[(X - m_{X|Y})(Y - m_Y)^T]$:

$$E[(X - m_{X|Y})(Y - m_Y)^T] = E[(X - m_X - R_{XY}R_Y^{-1}(Y - m_Y))(Y - m_Y)^T] =$$
$$E[(X - m_X)(Y - m_Y)^T] - R_{XY}R_Y^{-1}E[(Y - m_Y)(Y - m_Y)^T] = R_{XY} - R_{XY}R_Y^{-1}R_Y = 0,$$

and so $X - m_{X|Y}$ and Y are uncorrelated normally distributed stochastic variables and hence by (a) independent, too.

Remark 4.2 By (15) and (b) when X and Y are normally distributed s.v.s, the MMS estimate is an affine function of y! (A linear function if $m_X = 0$ and $m_Y = 0$.) It is said that the MMS estimator is then a *linear estimator*. In general, the optimal estimate depends nonlinearly on y.

A Useful formula

Let X, U and V be normally distributed vector-valued stochastic variables, and let U and V be independent. Then

$$E[X \mid U = u, V = v] = E[X \mid U = u] + E[X \mid V = v] - E[X]. \tag{27}$$

This we see as follows. Put

$$Y = \begin{pmatrix} U \\ V \end{pmatrix}.$$

It holds by independence of U and V that $R_{UV} = 0$ and thus

$$R_Y = \begin{pmatrix} R_U & 0 \\ 0 & R_V \end{pmatrix}.$$

Furthermore

$$R_{XY} = E[(X - m_X)(Y - m_Y)^T] =$$
$$E[(X - m_X)(U - m_U)^T, (V - m_V)^T] = [R_{XU} \quad R_{XV}]$$

and so

$$R_{XY}R_Y^{-1} = [R_{XU}, R_{XV}] \begin{pmatrix} R_U^{-1} & 0 \\ 0 & R_V^{-1} \end{pmatrix} = [R_{XU}R_U^{-1} \quad {}_{XV}R_V^{-1}].$$

Therefore by (19) and (20) in (b), it follows that

$$E[X \mid U = u, V = v] = E[X \mid Y = (u^T, v^T)^T] =$$
$$m_X + [R_{XU}R_U^{-1} \quad R_{XV}R_V^{-1}][(u^T, v^T)^T - (m_u^T, m_v^T)^T] =$$
$$m_X + R_{XU}R_U^{-1}(u - m_U) + R_{XV}R_V^{-1}(v - m_V) =$$
$$[m_X + R_{XU}R_U^{-1}(u - m_U)] + [m_X + R_{XV}R_V^{-1}(v - m_V)] - m_X =$$
$$E[X \mid U = u] + E[X \mid V = v] - m_X.$$

4.3 Linear Estimators

We shall now proceed to study so-called linear estimators as defined in remark 4.2. The MMS estimator is, in general, a nonlinear estimator. Hence it is, in general, hard to determine the MMS estimator due to the difficulty in evaluating the conditional mean $E[X \mid Y = y]$, except in the case of normally distributed stochastic variables.

Consider the problem of determining the optimal *linear estimate* (optimal in the mean square sense)

$$\hat{x} = \hat{x}(y) = Ay + b \tag{28}$$

which minimizes the mean square error

$$\ell \equiv E[(X - \hat{x})^T (X - \hat{x})]. \tag{29}$$

Note that we can not use the conditional mean square error (13) here as this would define the optimal values of A and b as functions of y, i.e. we would obtain a nonlinear estimate (estimator), in general. This is avoided when using (29) as the expectation is taken here over the joint distribution of X and Y.

We write, cf. (28),

$$AY + b = \begin{pmatrix} A & b \end{pmatrix} \begin{pmatrix} Y \\ 1 \end{pmatrix} = BZ,$$

where $B = (A \quad b)$ and

$$Z = \begin{pmatrix} Y \\ 1 \end{pmatrix} = (Y^T \quad 1)^T.$$

Then

$$\ell = E[(X - BZ)^T (X - BZ)] = E[X^T X] - 2E[X^T BX] + E[Z^T B^T BZ]. \tag{30}$$

Interlude: Trace function for matrices.

Let S be a square matrix (of size $n \times n$). The trace of S, $\operatorname{tr} S$, is defined as

$$\operatorname{tr} S = \sum_{i=1}^{n} S_{ii}.$$

Clearly then $\operatorname{tr} S^T = \operatorname{tr} S$. Let P be an $n \times m$ and T an $m \times n$ matrix. Then

$$\operatorname{tr}(PT) = \sum_{i=1}^{n} (\sum_{j=1}^{m} P_{ij} T_{ji}) = \sum_{j=1}^{m} \sum_{i=1}^{n} T_{ji} P_{ij} = \operatorname{tr}(TP).$$

Now back to obtained expression (30) for the mean square error ℓ. Using the trace function we can write (30) as

$$\begin{aligned} \ell &= E[X^T X] - 2E[\operatorname{tr}(BZX^T)] + E[\operatorname{tr}(BZZ^T B^T)] \\ &= E[X^T X] - 2\operatorname{tr}(BE[ZX^T]) + \operatorname{tr}(BE[ZZ^T]B^T) \end{aligned}$$

Then we complete squares and get

$$\ell = \mathrm{tr}\left\{ \left[BE[ZZ^T] - E[XZ^T]\right] (E[ZZ^T])^{-1} \left[BE[ZZ^T] - E[XZ^T]\right]^T \right\}$$
$$-\mathrm{tr}\left\{E[XZ^T](E[ZZ^T])^{-1}(E[XZ^T])^T\right\} + E[X^TX]$$

Note that here only the first trace expression depends on B, i.e. of the parameters with respect to which ℓ is to be minimized.

Let H be any matrix and let H_i denote the $i-$th column of H. As $\mathrm{tr}(H^TGH) = \sum_i H_i^TGH_i \geq 0$ for any positive definite (and symmetric) matrix G of compatible dimensions (so that the product H^TGH makes sense), it follows that ℓ is minimized by choosing B so that $BE[ZZ^T] - E[XZ^T] = 0$, i.e.

$$BE[ZZ^T] = E[XZ^T] \tag{31}$$

Here

$$E[ZZ^T] = E\left[\begin{pmatrix} Y \\ 1 \end{pmatrix} \begin{pmatrix} Y^T & 1 \end{pmatrix}\right] = \begin{pmatrix} E[YY^T] & E[Y] \\ (E[Y])^T & 1 \end{pmatrix}$$

and so

$$E[ZZ^T] = \begin{pmatrix} R_Y + m_Y m_Y^T & m_Y \\ m_Y^T & 1 \end{pmatrix}$$

as $R_Y = E[Y - m_Y)(Y - m_Y)^T] = E[YY^T] - m_Y m_Y^T$.

Furthermore

$$E[XZ^T] = E[X(Y^T \quad 1)] = (E[XY^T] \quad m_X).$$

We need to express $E[XY^T]$ using known quantities. So we compute

$$R_{XY} = E[(X - m_X)(Y - m_Y)^T] = E[XY^T] - m_X m_Y^T - m_X m_Y^T + m_X m_Y^T.$$

This gives that

$$E[XY^T] = R_{XY} + m_X m_Y^T.$$

Using the definition of B and the above expressions for $E[ZZ^T]$ and $E[XZ^T]$ gives that

$$\begin{pmatrix} A & b \end{pmatrix} \begin{pmatrix} R_Y + m_Y m_Y^T & m_Y \\ m_Y^T & 1 \end{pmatrix} = \begin{pmatrix} R_{XY} + m_X m_Y^T & m_X \end{pmatrix}$$

and so

$$\begin{aligned} A(R_Y + m_Y m_Y^T) + b m_Y^T &= R_{XY} + m_X m_Y^T \\ Am_Y + b &= m_X \end{aligned} \tag{32}$$

Inserting $b = m_X - Am_Y$ in (32) gives that $AR_Y = R_{XY}$. Thus the optimal values of A and b are

$$\begin{aligned} A &= R_{XY}R_Y^{-1} \\ b &= m_X - R_{XY}R_Y^{-1}m_Y \end{aligned}$$

Hence the optimal linear estimate (estimator) is given by

$$\hat{x} = Ay + b = m_X + R_{XY}R_Y^{-1}(y - m_Y) \tag{33}$$

The optimal linear estimate (33) is the same result that we obtained in section 4.2 (see equation (19)) as the optimal estimate over all linear and nonlinear estimators for normally distributed variables. In the general case, the estimate (33) is only the optimal linear estimate (in general

$$E[X \mid Y = y] \neq m_X + R_{XY}R_Y^{-1}(y - m_Y),$$

where the left-hand side is the gobally optimal estimate and the right-hand side is the optimal linear estimate).

Properties of the estimate (33)

For the estimate (33) the estimation error $X - \hat{x}$ satisfies

(i) $E[X - \hat{x}] = 0$ (zero mean value)
(ii) $E[(X - \hat{x})Y^T] = 0$ (uncorrelated with Y)
(iii) $E[(X - \hat{x})\hat{x}^T] = 0$ (uncorrelated with $\hat{x}$)

The terminology "uncorrelated with Y" in (ii) is justified as by (i) and (ii): $E[(X - \hat{x})(Y - m_Y)^T] = E[(X - \hat{x})Y^T] = 0$. Similarly in (iii), it follows from (i) and (iii) that $E[(X - \hat{x})(\hat{x} - m_{\hat{x}})^T = E[(X - \hat{x})\hat{x}^T] = 0$. Note that $m_{\hat{x}} = E[X] = m_X$ by (i).

Properties (i)-(iii) are established as follows. To derive (i), we compute

$$E[X - \hat{x}] = E[X] - m_X - R_{XY}R_Y^{-1}E[Y - m_Y] = 0$$

To establish (ii), we compute

$$E[(X - \hat{x})Y^T] = E[XY^T] - E[(m_X + R_{XY}R_Y^{-1}(Y - m_Y))Y^T] =$$
$$R_{XY} + m_X m_Y^T - m_X m_Y^T - R_{XY}R_Y^{-1}E[(Y - m_Y)Y^T] =$$
$$R_{XY} - R_{XY}R_Y^{-1}E[(Y - m_Y)(Y - m_Y + m_Y)^T] = R_{XY} - R_{XY}R_Y^{-1}R_Y - 0 = 0,$$

where we have used that $R_Y = E[(Y - m_Y)(Y - m_Y)^T]$ and $E[Y - m_Y] = 0$.

Finally, property (iii) follows directly by inserting (33) and using (i) and (ii).

For normally distributed variables (i)-(iii) follow directly from properties (a)-(c) in section 4.2: so property (i) holds as

$$E[X - \hat{x}] = E[X] - E_Y[E[X \mid Y]] = 0$$

by the law of total probability. By property (b) in section (4.2), we know that Y and $X - m_{X|Y} = X - \hat{x}$ are independent implying (ii) and (iii).

5 Optimal State Estimation – The Kalman Filter

In this section we consider optimal state estimation and derive the optimal estimator, which is called the Kalman filter. There are actually several minimum variance state estimation problems of interest depending on the information available to the estimator, corresponding to predictive and filtering forms of the Kalman filter.

5.1 The Predictive Kalman filter

Consider the state space system

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) + w(t) && (34)\\ y(t) &= Cx(t) + v(t) && (35) \end{aligned}$$

where $\{w(t)\}$ and $\{v(t)\}$ are sequences of independent *normally distributed* vectors (stochastic variables) with zero mean values and the covariance matrices

$$\begin{aligned} E[w(t)w(s)^T] &= R_1\delta_{t,s} \\ E[v(t)v(s)^T] &= R_2\delta_{t,s} && (36)\\ E[w(t)v(s)^T] &= 0 && (37) \end{aligned}$$

where $\delta_{t,s} = 1$ for $t = s$ and $\delta_{t,s} = 0$ for $t \neq s$ ($\delta_{t,s}$ is the so-called Kronecker delta). It is assumed that the initial state $x(t_0)$ is normally distributed with mean value m and covariance matrix R_0, and that the initial state is independent of $\{w(t)\}$ and $\{v(t)\}$.

Remark 5.1 We are here using somewhat stronger assumptions for the noise (normality!) than in the state space model (2)-(3) in section 3.1 (there the noise was assumed to be white), motivated by the result of the previous section that in the case of normal distributions, we can express explicitly the global minimum mean square (MMS) estimate!

We shall now consider state estimation for the system (34)-(35). We shall assume that at time t, the following information is available:

$$V_t = [y(t_0)^T, y(t_0+1)^T, \ldots, y(t)^T, u(t_0)^T, u(t_0+1)^T, \ldots, u(t)^T]^T.$$

We consider the problem of finding the estimate $\hat{x}$ of $x(t+1)$ based on the information V_t, so that the conditional mean square error

$$E[(x(t+1) - \hat{x})^T(x(t+1) - \hat{x}) \mid V_t]$$

is minimized. By the result (15) the MMS estimate is given by

$$\hat{x}(t+1 \mid t) = E[x(t+1) \mid V_t],$$

i.e. as the conditional mean of $x(t+1)$ given the information V_t. As we have assumed that all the stochastic variables are normally distributed, the conditional mean is given explicitly by (19). However, since the dimension of the vector V_t grows with time, it would be inefficient to apply (19) directly.

Can we find recursive expressions so that $\hat{x}(t+1 \mid t)$ is computed from $\hat{x}(t \mid t-1)$ when information about $y(t)$ and $u(t)$ is obtained? (It was Kalman's **BIG** contribution to show that this can indeed be done efficiently.)

At the initial time t_0, write

$$\hat{x}(t_0 \mid t_0 - 1) \equiv E[x(t_0)] = m$$

(We assume no information about y and u prior to the time t_0!) Then we proceed by induction, assuming that $\hat{x}(t \mid t-1)$ is given, and find an expression for $\hat{x}(t+1 \mid t)$.

At time t we have

$$\hat{x}(t+1 \mid t) = E[x(t+1) \mid V_t] = E[x(t+1) \mid V_{t-1}, y(t), u(t)]. \tag{38}$$

Now we shall use a property of the conditional mean as given in (27) – so this is a useful formula as claimed earlier! In order that this result can be used, the expectation must be taken with respect to stochastic variables which are independent. For this reason we introduce

$$\tilde{y}(t \mid t-1) = y(t) - E[y(t) \mid V_{t-1}] = y(t) - E[Cx(t) + v(t) \mid V_{t-1}],$$

where the last equality follows by (35). Suppose $v(t)$ is independent of $u(t-1)$, $u(t-2)$, ..., $u(t_0)$. (This holds for example if u is a given (deterministic) signal or if say $u(t) = Ky(t)$ or $u(t) = K_1 y(t) + K_2 y(t-1)$, i.e if $u(t)$ is given by causal feedback of $y(t)$, $y(t-1)$ and so on.) Then

$$\tilde{y}(t \mid t-1) = y(t) - E[Cx(t) \mid V_{t-1}] - E[v(t)] = y(t) - CE[x(t) \mid V_{t-1}] = y(t) - C\hat{x}(t \mid t-1).$$

Note that $y(t) = \tilde{y}(t \mid t-1) + C\hat{x}(t \mid t-1)$ and as $\hat{x}(t \mid t-1)$ is determined by V_{t-1}, we can replace in (38) V_t by V_{t-1}, $u(t)$, $\tilde{y}(t \mid t-1)$. (This gives the same information state!) Furthermore, we observe that

$$\tilde{y}(t \mid t-1) = C(x(t) - \hat{x}(t \mid t-1)) + v(t) = C\tilde{x}(t \mid t-1) + v(t),$$

where

$$\tilde{x}(t \mid t-1) = x(t) - \hat{x}(t \mid t-1)$$

denotes the estimation error. Hence as by property (c) in section 4.2, $\tilde{x}(t \mid t-1)$ is independent of V_{t-1}, so is $\tilde{y}(t \mid t-1)$. Therefore we can proceed from (38) to compute using (34)

$$\begin{aligned}
&\hat{x}(t+1 \mid t) = E[x(t+1) \mid V_{t-1}, u(t), \tilde{y}(t \mid t-1)] = \\
&E[Ax(t) + Bu(t) + w(t) \mid V_{t-1}, u(t), \tilde{y}(t \mid t-1)] = \\
&AE[x(t) \mid V_{t-1}, u(t), \tilde{y}(t \mid t-1)] + Bu(t) + E[w(t)]
\end{aligned}$$

as $u(t)$ is a known vector at time t (it belongs to the information state at time t) and $w(t)$ is independent of V_{t-1} and $\tilde{y}(t \mid t-1)$, and as $w(t)$ is also assumed to be independent of $u(t)$.

Finally, we use that $E[w(t)] = 0$ by assumption and so

$$\hat{x}(t+1 \mid t) = AE[x(t) \mid V_{t-1}, u(t), \tilde{y}(t \mid t-1)] + Bu(t). \tag{39}$$

The value of the above conditional expectation depends on what we assume about the input $u(t)$. We consider two cases. **Firstly**, if $u(t)$ is a function of V_{t-1} and $y(t)$ (or equivalently of V_{t-1} and $\tilde{y}(t \mid t-1)$), as in the case of causal input-output feedback, then

$$E[x(t) \mid V_{t-1}, u(t), \tilde{y}(t \mid t-1)] = E[x(t) \mid V_{t-1}, \tilde{y}(t \mid t-1)]$$

as the two indicated information states are then equivalent. In this case it therefore holds by (27) that (finally we can apply this formula!)

$$E[x(t \mid V_{t-1}, u(t), \tilde{y}(t \mid t-1)] = E[x(t) \mid V_{t-1}] + E[x(t) \mid \tilde{y}(t \mid t-1)] - E[x(t)]. \quad (40)$$

Secondly, if $\{u(t)\}$ is some given (deterministic) sequence or a realization of a stochastic process independent of $x(t_0)$, $\{w(t)\}$ and $\{v(t)\}$ then

$$E[x(t) \mid V_{t-1}, u(t), \tilde{y}(t \mid t-1)] = E[x(t) \mid V_{t-1}, u(t)] + E[x(t) \mid \tilde{y}(t \mid t-1)] - E[x(t)] = \\ E[x(t) \mid V_{t-1}] + E[x(t) \mid \tilde{y}(t \mid t-1)] - E[x(t)]$$

as we can again apply equation (27) and as $u(t)$ now contains no additional information about $x(t)$ compared to V_{t-1}. Note that the above obtained relationship is the same as (40).

Recalling the definition $\hat{x}(t \mid t-1) = E[x(t) \mid V_{t-1}]$ and inserting (40) into (39) gives finally that

$$\hat{x}(t+1 \mid t) = A\hat{x}(t \mid t-1) + Bu(t) + A\left(E[x(t) \mid \tilde{y}(t \mid t-1)] - E[x(t)]\right). \quad (41)$$

We need to evaluate $E[x(t) \mid \tilde{y}(t \mid t-1)]$. We use property (b) of normally distributed stochastic variables in section 4.2, i.e. the relationships (19)-(20). We need the covariance matrices $R_{x\tilde{y}}$ and $R_{\tilde{y}}$. So we compute

$$E[\tilde{y}(t \mid t-1)] = E[y(t)] - E[E[y(t) \mid V_{t-1}]] = E[y(t)] - E[y(t)] = 0$$

by the law of total probability. Thus

$$R_{x\tilde{y}} = E[(x(t) - E[x(t)])\tilde{y}(t \mid t-1)^T] = E\left[(x(t) - E[x(t)])\left(C\tilde{x}(t \mid t-1) + v(t)\right)^T\right] = \\ E[(x(t) - E[x(t)])\tilde{x}(t \mid t-1)^T]C^T$$

as $\{w(t)\}$ and $\{v(t)\}$ are mutually independent and $w(t)$ and $v(t)$ are independent of $x(t)$ and have zero means.

Now

$$E\left[(E[x(t)])\tilde{x}(t \mid t-1)^T\right] = E[x(t)]E[\tilde{x}(t \mid t-1)^T] = 0$$

since

$$E[\tilde{x}(t \mid t-1)] = E[x(t)] - E[E[x(t) \mid V_{t-1}]] = E[x(t)] - E[x(t)] = 0$$

by the law of total probability. So the earlier obtained extression for $R_{x\tilde{y}}$ simplifies to

$$R_{x\tilde{y}} = E[x(t) \mid \tilde{x}(t \mid t-1)^T]C^T.$$

Then

$$R_{x\tilde{y}} = E\left[(\hat{x}(t \mid t-1) + \tilde{x}(t \mid t-1))\,\tilde{x}(t \mid t-1)^T\right] C^T = E[\tilde{x}(t \mid t-1)\tilde{x}(t \mid t-1)^T]C^T$$

as by the properties of an optimal estimate

$$E[\hat{x}(t \mid t-1)\tilde{x}(t \mid t-1)^T] = 0,$$

see property (c) in section 4.2 and property (iii) of the optimal linear estimator in section 4.3, the optimal linear estimate being also the globally optimal estimate in the case of normally distributed stochastic variables.

Introduce the covariance matrix for the estimation error $\tilde{x}(t \mid t-1)$

$$P_x(t) = E[\tilde{x}(t \mid t-1)\tilde{x}(t \mid t-1)^T].$$

Then we can write $R_{x\tilde{y}}$ as

$$R_{x\tilde{y}} = P_x(t)C^T.$$

We proceed to evaluate $R_{\tilde{y}}$. We recall that $\tilde{y}(t \mid t-1) = C\tilde{x}(t \mid t-1) + v(t)$ and so $E[\tilde{y}(t \mid t-1)] = 0$. Thus

$$R_{\tilde{y}} = E[\tilde{y}(t \mid t-1)\tilde{y}(t \mid t-1)^T] = E\left[(C\tilde{x}(t \mid t-1) + v(t))\left((C\tilde{x}(t \mid t-1) + v(t)\right)^T\right] = CP_x(t)C^T + R_2$$

as $E[\tilde{x}(t \mid t-1)v(t)^T] = 0$. Now we are ready to apply property (b) in section 4.2 to obtain an expression for the term $E[x(t) \mid \tilde{y}(t \mid t-1)]$. Thus

$$E[x(t) \mid \tilde{y}(t \mid t-1)] = E[x(t)] + \bar{K}(t)\tilde{y}(t \mid t-1),$$

where

$$\bar{K}(t) = R_{x\tilde{y}}R_{\tilde{y}}^{-1} = P_x(t)C^T(CP_x(t)C^T + R_2)^{-1}.$$

We denote $K(t) = A\bar{K}(t)$ and so inserting all this into (41), gives finally the optimal estimate in recursive form as

$$\hat{x}(t+1 \mid t) = A\hat{x}(t \mid t-1) + Bu(t) + K(t)\tilde{y}(t \mid t-1), \tag{42}$$

where we recall that $\tilde{y}(t \mid t-1) = y(t) - C\hat{x}(t \mid t-1)$. The initial condition for the recursive equation (42) is recalled from our earlier assumption to be: $\hat{x}(t_0 \mid t_0 - 1) = m$. Note that here we can interpret the term $K(t)\tilde{y}(t \mid t-1)$ as a correction term due to the new measurement information $y(t)$.

It remains to evaluate the covariance matrix $P_x(t)$ recursively. Equations (34) and (42) give that

$$\begin{aligned}
\tilde{x}(t+1) \mid t) &= x(t+1) - \hat{x}(t+1 \mid t) \\
&= A[x(t) - \tilde{x}(t \mid t-1)] + w(t) - K(t)\tilde{y}(t \mid t-1) \\
&= A\tilde{x}(t \mid t-1) + w(t) - K(t)[C\tilde{x}(t \mid t-1) + v(t)] \\
&= (A - K(t)C)\tilde{x}(t \mid t-1) + w(t) - K(t)v(t).
\end{aligned}$$

This gives that $E[\tilde{x}(t+1 \mid t)] = 0$ as $E[\tilde{x}(t \mid t-1) = 0$ (and as $E[w(t)] = 0$ and $E[v(t)] = 0$). Hence

$$P_x(t+1) = (A - K(t)C)P_x(t)(A - K(t)C)^T + R_1 + K(t)R_2K(t)^T$$

as $E[w(t)v(t)^T] = 0$. We write the above relationship in the form

$$P_x(t+1) = AP_x(t)A^T + K(T)(CP_x(t)C^T + R_2)K(t)^T - AP_x(t)C^TK(t)^T - K(t)CP_x(t)A^T + R_1$$

and replace here the first $K(t)$ term with our earlier expression for $K(t) = A\bar{K}(t) = AP_x(t)C^T(CP_x(t)C^T + R_2)^{-1}$. This gives

$$\begin{aligned} P_x(t+1) &= AP_x(t)A^T + AP_x(t)C^TK(t)^T - AP_x(t)C^TK(t)^T - K(t)CP_x(t)A^T + R_1 \\ &= AP_x(t)A^T - K(t)CP_x(t)A^T + R_1. \end{aligned}$$

Finally we insert here the expression defining $K(t)$ to get

$$P_x(t+1) = AP_x(t)A^T - AP_x(t)C^T\left(CP_x(t)C^T + R_2\right)^{-1}CP_x(t)A^T + R_1.$$

As these recursive equations for the optimal predictive estimate (due to Rudolf Kalman) are so important, we summarize the obtained results:

Summary: Kalman filter – predictive case

The minimum mean square (MMS) estimate of $x(t+1)$ for the system (34)-(35) given the information $\{y(t_0), y(t_0+1), \ldots, y(t), u(t_0), u(t_0+1), \ldots, u(t)\}$ is the conditional expectation $\hat{x}(t+1 \mid t)$, which can be computed recursively according to the Kalman filter equation

$$\begin{aligned} \hat{x}(t+1 \mid t) &= A\hat{x}(t \mid t-1) + Bu(t) + K(t)(y(t) - C\hat{x}(t \mid t-1)) \quad (43) \\ \hat{x}(t_0 \mid t_0 - 1) &= m, \quad (44) \end{aligned}$$

where m is the mean value of $x(t_0)$. The (Kalman filter) gain $K(t)$ is given by

$$K(t) = AP_x(t)C^T\left(CP_x(t)C^T + R_2\right)^{-1},$$

where $P_x(t)$ is the covariance matrix of the estimation error. This is given recursively by the Riccati equation

$$P_x(t+1) = AP_x(t)A^T - AP_x(t)C^T\left(CP_x(t)C^T + R_2\right)^{-1}CP_x(t)A^T + R_1 \quad (45)$$

with the initial value

$$P_x(t_0) = R_0,$$

where R_0 is the covariance matrix of $x(t_0)$.

Remark 5.2 Note that $P_x(t)$ does not depend on the observations $\{y(t_0), \ldots, y(t)\}$ nor on $\{u(t_0), \ldots, u(t)\}$. Hence $P_x(t)$ and the Kalman filter gain $K(t)$ can be computed *a priori* (off-line)!

Remark 5.3 As the derivation of the Kalman filter above did not use time invariance (constancy) of A, B and C, the result is also valid when (34)-(35) presents a time-varying system

$$\begin{aligned} x(t+1) &= A(t)x(t) + B(t)u(t) + w(t) \\ y(t) &= C(t)x(t) + v(t) \end{aligned}$$

with nonstationary noise $w(t)$ and $v(t)$ (having possibly time-varying covariance matrices $R_1(t)$ and $R_2(t)$, respectively).

Equation (45) is a discrete Riccati equation. Under appropriate conditions the solution $P_x(t)$ converges as $t \to \infty$. The matrix

$$P_x = \lim_{t\to\infty} P_x(t)$$

is then the stationary covariance matrix of the estimation error, and satisfies then the stationary (or *algebraic*) *discrete Riccati equation*

$$P_x = AP_xA^T - AP_xC^T\left(CP_xC^T + R_2\right)^{-1}CP_xA^T + R_1. \tag{46}$$

The filter gain converges then to the value

$$K = \lim_{t\to\infty} K(t) = AP_xC^T\left(CP_xC^T + R_2\right)^{-1}.$$

Equation (46) is a nonlinear matrix equation in P_x.

From the result (33) and the derivation of the optimal filter for the state estimation problem, it follows that if the disturbances in (34)-(35) are *not normally distributed*, the filter (43)-(45) is still the *optimal linear filter* for the estimation of $x(t+1)$ based on the information $\{y(t_0), \ldots, y(t), u(t_0), \ldots, u(t)\}$.

Example 10 We consider the DARE – Discrete Algebraic Riccati Equation – in the scalar case (the dimension of the state, the input and the output are all equal to 1). Then (46) can be written as (using lowercase letters as symbols for the corresponding matrices in (46))

$$p_x = ap_xa - ap_xc(cp_xc + r_2)^{-1}cp_xa + r_1.$$

This gives

$$(cp_xc + r_2)(1 - a^2)p_x = (cp_xc + r_2)r_1 - a^2p_x^2c^2$$

and so

$$\left((1-a^2)c^2 + a^2c^2\right)p_x^2 + \left((1-a^2)r_2 - c^2r_1\right)p_x - r_1r_2 = 0.$$

This can be written as

$$c^2p_x^2 + \left((1-a^2)r_2 - c^2r_1\right)p_x - r_1r_2 = 0.$$

This second-order equation in p_x can be solved to give

$$p_x = \frac{-\left((1-a^2)r_2 - c^2 r_1\right) \pm \sqrt{\left((1-a^2)r_2 - c^2 r_1\right)^2 + 4c^2 r_1 r_2}}{2c^2}$$

This second-order equation has only one nonegative solution $p_x \geq 0$ (we know that the limiting covariance matrix P_x must be a positive semidefinite, symmetric matrix (i.e. $P_x \geq 0$) when it exists, as $P_x(t)$ is such a matrix for all t):

$$p_x = \frac{-\left((1-a^2)r_2 - c^2 r_1\right) + \sqrt{\left((1-a^2)r_2 - c^2 r_1\right)^2 + 4c^2 r_1 r_2}}{2c^2}$$

This is the desired solution of the DARE.

Connection with Difference Equation Representation

In the *stationary* case ($t \to \infty$) (43) becomes

$$\begin{aligned} \hat{x}(t+1 \mid t) &= A\hat{x}(t \mid t-1) + Bu(t) + Ke(t) \\ y(t) &= C\hat{x}(t \mid t-1) + e(t), \end{aligned}$$

where $\{e(t)\}$ is a sequence of independent normally distributed vectors with covariance matrix

$$R_e = E[e(t)e(t)^T] = CP_xC^T + R_2.$$

(Recall that $\tilde{y}(t \mid t-1) = y(t) - C\hat{x}(t \mid t-1) = C\tilde{x}(t \mid t-1) + v(t)$. Thus as $\tilde{x}(t \mid t-1)$ approaches a stationary process with covariance matrix P_x when $t \to \infty$ and as $\tilde{x}(t \mid t-1)$ is independent of $v(t)$, it follows that $\tilde{y}(t \mid t-1)$ approaches a stationary process $\{e(t)\}$ with covariance matrix $CP_xC^T + R_2$.)

We can determine the transfer functions from u and e to y for the above stationary form of the predictive Kalman filter in the usual manner. This results in

$$y(t) = G_{yu}(q^{-1})u(t) + G_{ye}e(t),$$

where q^{-1} is the backwards (time) shift operator, i.e. $q^{-1}u(t) = u(t-1)$ and so on. Alternatively, this can be written in difference equation form as

$$y(t)+A_1y(t-1)+\ldots+A_ny(t-n) = B_0u(t-1)+\ldots+B_{n-1}u(t-n)+e(t)+C_1e(t-1)+C_ne(t-n).$$

This form is called the ARMAX representation (ARMAX= autoregressive moving average with an external input), where the y part is the AR part, the e part is the MA part and the u part is the X part. This representation is often used in system identification as system model structure.

5.2 Generalizations – Correlated Disturbances and the Filtering Form of the Kalman Filter

There are several important generalizations of the Kalman filter result that was presented earlier. We take up two of them.

Case of Correlated Disturbances

Consider the state space system

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) + w(t) \\ y(t) &= Cx(t) + v(t), \end{aligned}$$

where $\{w(t)\}$ and $\{v(t)\}$ are sequences of mutually correlated normally distributed stochastic vectors with zero mean values and the covariances

$$\begin{aligned} E[w(t)w(s)^T] &= \delta_{t,s}R_1 \\ E[v(t)v(s)^T] &= \delta_{t,s}R_2 \\ E[w(t)v(s)^T] &= \delta_{t,s}R_{12} \end{aligned}$$

The minimum mean square estimate $\hat{x}(t+1 \mid t) = E[x(t+1) \mid V_t]$ of $x(t+1)$ based on the information V_t, is then given by

$$\hat{x}(t+1 \mid t) = A\hat{x}(t \mid t-1) + Bu(t) + K(t)(y(t) - C\hat{x}(t \mid t-1)),$$

where

$$K(t) = \left(AP_x(t)C^T + R_{12}\right)\left(CP_x(t)C^T + R_2\right)^{-1}$$

and the covariance matrix $P_x(t)$ of the estimation error $\tilde{x}(t \mid t-1) = x(t) - \hat{x}(t \mid t-1)$ is given by

$$P_x(t+1) = AP_x(t)A^T - \left(AP_x(t)C^T + R_{12}\right)\left(CP_x(t)C^T + R_2\right)^{-1}\left(CP_x(t)A^T + R_{12}^T\right) + R_1. \tag{47}$$

This we see as follows. Note that earlier we obtained the predictive Kalman filter result when $R_{12} = 0$. Comparing with the derivation of (39), we see that

$$\hat{x}(t+1 \mid t) = AE[x(t) \mid V_{t-1}, u(t), \tilde{y}(t \mid t-1)] + Bu(t) + E[w(t) \mid V_{t-1}, u(t), \tilde{y}(t \mid t-1)].$$

(When $R_{12} = 0$ we had earlier that the conditional mean of $w(t)$ is equal to $E[w(t)] = 0$.) Now (using (27))

$$E[w(t) \mid V_{t-1}, u(t), \tilde{y}(t \mid t-1)] = E[w(t) \mid V_{t-1}] + E[w(t) \mid \tilde{y}(t \mid t-1)] - E[w(t)] = E[w(t) \mid \tilde{y}(t \mid t-1)],$$

as $E[w(t) \mid V_{t-1}] = E[w(t)] = 0$. This equation is valid under the same assumptions as used earlier for $u(t)$. As $w(t)$ and $\tilde{y}(t \mid t-1)$ are jointly normally distributed, we can

evaluate the above conditional mean as in (19). For this we need the covariance matrix $R_{w\tilde{y}}$ of $w(t)$ and $\tilde{y}(t \mid t-1)$. Now

$$R_{w\tilde{y}} = E[w(t)\tilde{y}(t \mid t-1)^T] = E[w(t)v(t)^T] = R_{12}.$$

Hence by (19)

$$E[w(t) \mid \tilde{y}(t \mid t-1)] = E[w(t)] + R_{12}R_{\tilde{y}}^{-1}\left(\tilde{y}(t \mid t-1) - E[\tilde{y}(t \mid t-1)\right) = R_{12}\left(CP_x(t)C^T + R_2\right)^{-1}\tilde{y}(t \mid t-1).$$

So the Kalman filter gain $K(t)$ is modified to

$$K(t) = \left(AP_x(t)C^T + R_{12}\right)\left(CP_x(t)C^T + R_{12}\right)^{-1}$$

and the recursive formula for the optimal estimate is then of the same form as earlier

$$\hat{x}(t+1 \mid t) = A\hat{x}(t \mid t-1) + Bu(t) + K(t)\tilde{y}(t \mid t-1).$$

We still need to consider how $P_x(t)$ changes when $R_{12} \neq 0$. Note that

$$\tilde{x}(t+1 \mid t) = (A - K(t)C)\tilde{x}(t \mid t-1) + w(t) - K(t)v(t),$$

i.e. the prediction error satisfies here an equation of the same form as earlier. This gives that

$$P_x(t+1) = (A - K(t)C)P_x(t)(A - K(t)C)^T + R_1 + K(t)R_2K(t)^T - R_{12}K(t)^T - K(t)R_{12}^T.$$

This can be simplified to (47) using the expression for $K(t)$.

Case of Filtering Form of the Kalman Filter

Often in applications, especially when the state estimate $\hat{x}$ is used in feedback control, it is important to consider the case when the optimal estimate of $x(t)$ uses the measured output at time t, i.e. $y(t)$, too.

Consider the state space system (34)-(35). The optimal estimate $\hat{x}(t+1 \mid t+1)$ of $x(t+1)$ based on the information V_{t+1} is given by the MMS estimate

$$\hat{x}(t+1 \mid t+1) = E[x(t+1) \mid V_{t+1}].$$

We shall now assume that either 1) we know nothing else about $\{u(t)\}$ other than its role in (34)-(35) as a given, fixed sequence or 2) that it is generated so that in cases 1) and 2), it is true that

$$E[x(t+1) \mid V_{t+1}] = E[x(t+1 \mid V_t, y(t+1)].$$

We have

$$\hat{x}(t+1 \mid t+1) = E[x(t+1) \mid V_t, y(t+1)] = E[x(t+1) \mid V_t, \tilde{y}(t+1 \mid t)],$$

where $\tilde{y}(t+1 \mid t) = y(t+1) - \hat{y}(t+1 \mid t)$. Now due to the properties of the optimal estimates, we recall from earlier that V_t and $\tilde{y}(t+1 \mid t)$ are independent normally distributed stochastic variables. Hence we can apply equation (27) to compute

$$\hat{x}(t+1 \mid t+1) = E[x(t+1) \mid V_t] + E[x(t+1) \mid \tilde{y}(t+1 \mid t)] - E[x(t+1)] = \hat{x}(t+1 \mid t) + \bar{K}(t+1)\tilde{y}(t+1 \mid t),$$

where we have also used an earlier computation leading to (42) and

$$\bar{K}(t+1) = P_x(t+1 \mid t)C^T \left(CP_x(t+1 \mid t)C^T + R_2\right)^{-1}.$$

Here $P_x(t+1 \mid \cdot)$ denotes the covariance matrix of the estimation error for the optimal estimate $\hat{x}(t+1 \mid \cdot)$ of $x(t+1)$. (So $P_x(t+1 \mid t)$ denotes $P_x(t)$ from the predictive estimate case and $P_x(t+1 \mid t+1)$ denotes the estimation error for $\hat{x}(t+1 \mid t+1)$ to estimate $x(t+1)$.)

Note that

$$\hat{x}(t+1 \mid t) = E[x(t+1) \mid V_t] = AE[x(t) \mid V_t] + Bu(t) + E[w(t) \mid V_t] = A\hat{x}(t \mid t) + Bu(t) + E[w(t)] = A\hat{x}(t \mid t) + Bu(t).$$

We still want to express $P_x(t+1 \mid t+1)$. Now

$$\begin{aligned}
\tilde{x}(t+1 \mid t+1) &= x(t+1) - \hat{x}(t+1 \mid t+1) \\
&= x(t+1) - \hat{x}(t+1 \mid t) - \bar{K}(t+1)\tilde{y}(t+1 \mid t) \\
&= \tilde{x}(t+1 \mid t) - \bar{K}(t+1)\tilde{y}(t+1 \mid t) = \\
&= \tilde{x}(t+1 \mid t) - \bar{K}(t+1)\left(C\tilde{x}(t+1 \mid t) + v(t+1)\right).
\end{aligned}$$

Hence

$$P_x(t+1 \mid t+1) = (I - \bar{K}(t+1)C)P_x(t+1 \mid t)(I - \bar{K}(t+1)C)^T + \bar{K}(t+1)R_2\bar{K}(t+1)^T$$

and so

$$P_x(t+1 \mid t+1) = P_x(t+1 \mid t) - \bar{K}(t+1)CP_x(t+1 \mid t) - P_x(t+1 \mid t)C^T\bar{K}(t+1)^T + \bar{K}(t+1)CP_x(t+1 \mid t)C^T\bar{K}(t+1)^T + \bar{K}(t+1)R_2\bar{K}(t+1)^T$$

Here the sum of the last two terms can be written as

$$\bar{K}(t+1)CP_x(t+1 \mid t)C^T\bar{K}(t+1)^T + \bar{K}(t+1)R_2\bar{K}(t+1)^T = \bar{K}(t+1)\left(CP_x(t+1 \mid t)C^T + R_2\right)\bar{K}(t+1)^T = P_x(t+1 \mid t)C^T\bar{K}(t+1)^T,$$

where we have used the defining expression for $\bar{K}(t+1)$. Therefore

$$P_x(t+1 \mid t+1) = P_x(t+1 \mid t) - \bar{K}(t+1)CP_x(t+1 \mid t) = P_x(t+1 \mid t) - P_x(t+1 \mid t)C^T\left(CP_x(t+1 \mid t)C^T + R_2\right)^{-1}CP_x(t+1 \mid t).$$

We see also that as

$$\tilde{x}(t+1 \mid t) = x(t+1) - \hat{x}(t+1 \mid t) = A(x(t) - \hat{x}(t \mid t)) + w(t),$$

it follows that

$$P_x(t+1 \mid t) = AP_x(t \mid t)A^T + R_1.$$

We summarize these important equations.

Summary: Kalman filter – filtering case

The minimum mean square estimate of $x(t+1)$ for the state space system (34)-(35), given the information $\{y(t_0), y(t_0+1), \ldots, y(t+1), u(t_0), \ldots, u(t)\}$, is the conditional expectation $\hat{x}(t+1 \mid t+1)$ which can be computed recursively according to the Kalman filter equation

$$\hat{x}(t+1 \mid t+1) = A\hat{x}(t \mid t) + Bu(t) + \bar{K}(t+1)\left(y(t+1) - C(A\hat{x}(t \mid t) + Bu(t)\right) \quad (48)$$

with the initial value $\hat{x}(t_0 \mid t_0) = m$. Furthermore

$$\bar{K}(t+1) = P_x(t+1 \mid t)C^T\left(CP_x(t+1 \mid t)C^T + R_2\right)^{-1},$$

where

$$P_x(t+1 \mid t) = AP_x(t \mid t)A^T + R_1$$

and

$$P_x(t+1 \mid t+1) = P_x(t+1 \mid t) - P_x(t+1 \mid t)C^T\left(CP_x(t+1 \mid t)C^T + R_2\right)^{-1}CP_x(t+1 \mid t)$$

with the initial value $P_x(t_0 \mid t_0) = R_0$.

Let us now consider the stationary case ($t \to \infty$) of the filtering form of the Kalman filter. Denote

$$\begin{aligned} P_{x,f} &= \lim_{t\to\infty} P_x(t \mid t) \\ P_{x,p} &= \lim_{t\to\infty} P_x(t \mid t-1) \end{aligned}$$

(Note that we have earlier denoted, in the context of the stationary case of the predictive Kalman filter, $P_{x,p}$ as P_x.)

Then

$$\begin{aligned} P_{x,p} &= AP_{x,f}A^T + R_1 \\ P_{x,f} &= P_{x,p} - P_{x,p}C^T\left(CP_{x,p}C^T + R_2\right)^{-1}CP_{x,p}. \end{aligned}$$

Denoting $H = P_{x,p}C^T\left(CP_{x,p}C^T + R_2\right)^{-1}CP_{x,p}$, we see that

$$v^TP_{x,f}v = v^TP_{x,p}v - v^THv \leq v^TP_{x,p}vquad\text{for all } v \in \mathbb{R}^n$$

as H is a symmetric, positive semidefinite matrix. Thus we can write compactly

$$P_{x,f} \leq P_{x,p},$$

i.e. the stationary covariance matrix of the estimation error for the filtering estimate of $x(t+1)$ is smaller than or equal to (in the precise sense as derived above), the stationary covariance matrix of the estimation error for the predictive estimate of $x(t+1)$. (As of course we would expect!)

Denote $\bar{K} = \lim_{t\to\infty} \bar{K}(t)$. The stationary form of (48) becomes then

$$\hat{x}(t+1 \mid t+1) = A\hat{x}(t \mid t) + Bu(t) + \bar{K}\left(y(t+1) - C[A\hat{x}(t \mid t) + Bu(t)]\right),$$

where

$$\bar{K} = P_{x,p}C^T\left(CP_{x,p}C^T + R_2\right)^{-1}$$

and $P_{x,p} = P_x$ is given as the solution to the discrete algebraic Riccati equation (DARE) (46).

Note that $P_{x,p}$ (and $P_{x,f}$) and $\bar{K}$, the Kalman filter gain, can be computed off-line!

Example 11 We verify in the scalar scase the (matrix) inequality $P_{x,f} \leq P_{x,p}$. (Lower case letters are used.) By our earlier computations

$$p_{x,f} = p_{x,p} - p_{x,p}c(cp_{x,p}c + r_2)^{-1}cp_{x,p},$$

and so

$$p_{x,f} = \left(1 - \frac{p_{x,p}c^2}{p_{x,p}c^2 + r_2}\right)p_{x,p} \leq p_{x,p}.$$

Thus indeed, it holds that $0 \leq p_{p,f} \leq p_{x,p}$.

6 On Some Practical Issues in Kalman Filtering

We shall here discuss several things to take into account in state estimation and in Kalman filtering in particular.

6.1 Unstable Systems

The steady-state predictive Kalman filter is of the form

$$\hat{x}(t+1) = (A - KC)\hat{x}(t) + Bu(t) + Ky(t)$$

for the state space model

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) + w(t) \\ y(t) &= Cx(t) + v(t) \end{aligned}$$

Denoting the prediction error as $\tilde{x}(t) = x(t) - \hat{x}(t)$, we get that

$$\tilde{x}(t+1) = (A - KC)\tilde{x}(t) + w(t) - Kv(t).$$

This would seem to indicate that as long as $A - KC$ is a stable matrix (i.e. all eigenvalues of the matrix are strictly less than one in magnitude), the estimation error dynamics is stable – even if A is unstable (i.e. even if the state space system being estimated is an unstable system).

However, in practice there would be a big problem! To see this consider that the data $\{y(t)\}$ is in reality generatated by the perturbed system

$$\begin{aligned} x(t+1) &= A'x(t) + Bu(t) + w(t) \\ y(t) &= Cx(t) + v(t), \end{aligned}$$

whilst the predictor is still given as before (i.e. A is the system matrix used in the predictor). So A' is a perturbation of the matrix A in the predictor. Then

$$\begin{gathered} \tilde{x}(t+1) = x(t+1) - \hat{x}(t+1) = A'x(t) - A\hat{x}(t) + KC\hat{x}(t) - Ky(t) + w(t) = \\ (A' - A)x(t) + A(x(t) - \hat{x}(t)) + KC(\hat{x}(t) - x(t)) - Kv(t) + w(t) = \\ (A' - A)x(t) + (A - KC)\tilde{x}(t) + w(t) - Kv(t), \end{gathered} \tag{49}$$

and so

$$\begin{pmatrix} x(t+1) \\ \tilde{x}(t+1) \end{pmatrix} = \begin{pmatrix} A' & 0 \\ A' - A & A - KC \end{pmatrix} \begin{pmatrix} x(t) \\ \tilde{x}(t) \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} u(t) + \begin{pmatrix} w(t) \\ w(t) - Kv(t) \end{pmatrix}.$$

If A' and A are both unstable and $A' \neq A$, then clearly even if $\max_{1 \leq i,j \leq n} |a'_{ij} - a_{ij}|$ is small, it is expected that $\tilde{x}(t)$ may become arbitrarily large when t grows (unstable error dynamics)! (Here $A' = [a'_{ij}]$ and $A = [a_{ij}]$.)

Example 12 Consider a scalar example. Take $A' = 2$, $A = 2 - \epsilon$, $\epsilon > 0$, and $A - KC = 0.5$. Set $u(t) \equiv 0$, $w(t) \equiv 0$ and $v(t) \equiv 0$ (just for the sake of easy of illustration). Then by (49)

$$\begin{pmatrix} x(t+1) \\ \tilde{x}(t+1) \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ \epsilon & 0.5 \end{pmatrix} \begin{pmatrix} x(t) \\ \tilde{x}(t) \end{pmatrix}.$$

Put $x(0) = 1$ and $\tilde{x}(0) = 0$. Then $x(1) = 2$, $x(2) = 4$ and $x(k) = 2^k$ for any $k \geq 1$. Clearly

$$\tilde{x}(k+1) \geq \epsilon \cdot 2^k \to \infty \quad \text{when } k \to \infty$$

however small $\epsilon > 0$ is!

Here we observe that the estimation error dynamics has zero robustness against perturbations in the system matrix, as the true system is unstable.

The zero robustness problem occurs when an unstable system has not been stabilized by feedback. Assume now that

$$u(t) = -Fy(t)$$

is a stabilizing output feedback law, i.e. that both $A - BFC$ and $A' - BFC$ are stable matrices (these matrices have thus all their eigenvalues strictly inside the unit circle). Then

$$\begin{pmatrix} x(t+1) \\ \tilde{x}(t+1) \end{pmatrix} = \begin{pmatrix} A' - BFC & 0 \\ A' - A & A - KC \end{pmatrix} \begin{pmatrix} x(t) \\ \tilde{x}(t) \end{pmatrix} + \begin{pmatrix} w(t) - BFv(t) \\ w(t) - Kv(t) \end{pmatrix}$$

and thus the zero robustness problem disappears (because if $A' - A$ is close to a zero matrix then the above state space system is stable and so also the error dynamics is stable – obviously this requires that $A - KC$ is a stable matrix).

In linear quadratic gaussian (LQG) control the stabilizing feedback law is taken to be of the form

$$u(t) = -L\hat{x}(t),$$

and if possible the state estimate $\hat{x}(t)$ should actually be taken to be the filtered estimate $\hat{x}(t \mid t)$, not the predictive estimate $\hat{x}(t \mid t-1)$ (the latter estimate gives a feedback law that does not feed back the most recent measurement $y(t)$).

Note that although we used in the above analysis the predictive filter, the zero robustness problem is also present for filters in the filtered form (48).

6.2 Numerical Issues

A full treatment of numerical issues is beyond the scope of the present course, but we shall point out several issues of relevance in the implementation of Kalman filters and recursive filters in general.

We shall mostly use here the predictive form of the Kalman filter in the discussion, but similar observations hold also for the filtering form of the Kalman filter.

The most basic numerical issue applies already to the steady-state Kalman filter, which does not involve any filter gain and error covariance matrix recursions. This steady-state filter is of the recursive form

$$\hat{x}(t+1) = A_K\hat{x}(t) + Ky(t).$$

(Note that we have dropped here the term $Bu(t)$ as it has no relevance for our discussion when $u(t)$ is a known deterministic signal.)

We shall assume that the measured signal $\{y(t)\}$ has been generated by some stable stochastic state space system and that A_K is a stable matrix. This would be a set of mild assumptions, which are expected to be realistic in most applications of Kalman filtering (especially for signal estimation purposes).

Let (for the sake of discussion) z denote the important signal (possibly vector-valued) that we really need to estimate and let

$$z = Dx,$$

so that the (optimal predictive) MMS estimate of $z(t)$ is given by

$$\hat{z}(t) = D\hat{x}(t).$$

The complete filter equations are

$$\begin{aligned} \hat{x}(t+1) &= A_K\hat{x}(t) + Ky(t) \\ \hat{z}(t) &= D\hat{x}(t). \end{aligned}$$

Introduce the variable transformation $\varphi(t) = S\hat{x}(t)$, where S is an invertible (square) matrix. The matrix S presents a coordinate transformation resulting in

$$\begin{aligned} \varphi(t+1) &= \left(SA_KS^{-1}\right)\varphi(t) + (SK)y(t) \\ \hat{z}(t) &= DS^{-1}\varphi(t) \end{aligned}$$

or

$$\begin{aligned} \varphi(t+1) &= T\varphi(t) + Ly(t) \\ \hat{z}(t) &= R\varphi(t), \end{aligned}$$

where $T = SA_KS^{-1}$, $L = SK$, and $R = DS^{-1}$.

The point we want to make here is that although T has the same eigenvalues as A_K in infinite precision arithmetics, there can be a huge difference in their eigenvalues and other characteristics when the filter is implemented in finite accuracy floating point or fixed point microprocessors, digital signal processors or microcontrollers!

The general non-steady-state predictive Kalman filter involves also filter gain updating and estimation error covariance matrix updating. These quantities do not, however, depend on the measurements $\{y(t)\}$ and can therefore be computed off-line and stored in some approximate function interpolation form to save memory requirements. In any case this complicates the filter computations in comparison with the simple steady-state filter.

A practical issue is that the computed error covariance matrix $P_x(t)$ could lose its symmetry and positive semidefiniteness properties due to error propagation in numerical computations. Hence it is often recommended that in stead of $P_x(t)$, it is better to update some stable matrix factors of a numerically robust factorization of the error covariance matrix. This will unfortunately result in a more complicated algorithm.

6.3 Some References on Kalman filtering

We refer to the references given in the Introduction section of these lectures notes (Part I – Kalman Filtering). The following additional information should be useful.

A good general reference on Kalman filtering is the classic book: K.J. Åström, Introduction to Stochastic Control Theory, Academic Press, 1970. This reference treats both the discrete time and the continuous time case. The book by Torsten Söderström (Discrete-time Stochastic Systems: Estimation & Control, Prentice Hall, 1994) is nicely written and covers similar topics as the present lecture notes and more.

Another nice reference is the following book: G.C. Goodwin and K.S. Sin, Adaptive Filtering, Prediction and Control, Prentice-Hall, 1984.

A reference that treats many implementation and numerical issues as well as the nonlinear case is the book: M.S. Grewal and A.P. Andrews, Kalman Filtering: Theory and Practice, Prentice Hall, 1993.

Several popular techniques in recursive parameter estimation and in adaptive filtering can be interpreted as Kalman filtering methods. Some of these connections are discussed in the book: L. Ljung and T. Söderström, Theory and Practice of Recursive Identification, The MIT Press, 1983.

Finally, a fairly recent article dealing with an interesting application of Kalman filtering to navigation is as follows: J. Farrell and T. Givargis, Differential GPS reference station algorithm – Design and analysis. IEEE Trans. Control Syst. Technology, vol. 8, no. 3, pp. 519–531, May 2000.