Kalman Filtering and Linear Quadratic Gaussian Control

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PART II - Linear Quadratic Gaussian Control

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Contents

1 A Brief Introduction to Part II

This Part II of the lecture notes for course 7604120 is a continuation of Part I of the lecture notes. Part II deals with Linear Quadratic Gaussian (LQG) control of stochastic state space systems. The solution of optimal LQG control problems is closely associated with optimal state estimation, i.e. with Kalman filtering, a topic that was studied in detail in Part I of these lecture notes.

We do not repeat here the background material and historical remarks concerning LQG control made in the Introduction of Part I of these lecture notes. Instead we start directly with dynamic programming, a fundamental approach in optimization theory, which is instrumental in the solution of LQG control problems.

2 Dynamic Programming

In this section we study dynamic programming and its application to stochastic optimization problems.

2.1 Principle of Optimality and Bellman Equation

Dynamic programming is a mathematical technique for solving sequential decision and optimization problems. It was developed by Richard Bellman and his associates at the Rand Corporation in the 1950s and it is especially important in stochastic optimization problems.

Dynamic programming is based on the principle of optimality allowing the solution of sequential optimization/decision problems in a recursive manner. An optimal strategy or an optimizer solving the optimization/decision problem is called an optimal policy.

Principle of Optimality: An optimal policy has the property that, whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with respect to the state resulting from the first decision.

Example 1 A positive quantity c is to be divided into n parts in such a way that the product of the n parts is to be a maximum. Use recursion to obtain the optimal subdivision.

Solution: Let $f_n(c)$ denote the maximum attainable product as a function of c and n. If we regard c as fixed, and let n vary over the positive integers, $f_n(c)$ becomes a function of the integer variable n. Define $f_1(c) = c$. For $n = 2$ it holds that

$$f_2(c) = \max_{0 \leq y \leq c} y f_1(c-y) = \max_{0 \leq y \leq c} y(c-y) = \frac{c^2}{4}$$

for $y = c/2$. (This we can check by noting that $h(y) = y(c - y)$ and so $h'(y) = c - 2y$. Clearly then h has a unique global maximum at $y^* = c/2$ $(0 \leq y^* \leq c)$ and $h(y^*) = c^2/4$.) By optimal policy we mean the optimal subdivision, which is here

$$\begin{array}{lll} \text{Optimal} & \text{policy} & (c/2, c/2) \\ \text{Optimal} & \text{value} & f_2(c) = c^2/4 \end{array}$$

Suppose that we have solved the problem (that is we know $f_n(c)$) for $n(\geq 2)$ and wish to obtain the solution for $n+1$. (As we have solved the problem for $n = 2$, this would then constitute an inductive or recursive solution of the problem for any $n \geq 2$.)

Let the first of the $n+1$ parts be denoted as y. We have then $c - y$ (as $0 \leq y \leq c$) to be divided into n further parts. The maximum value $f_{n+1}(c)$ is given by

$$f_{n+1}(c) = \max_{0 \leq y \leq c} y f_n(c - y),$$

where we have used the fact that the **Principle of Optimality** can be used.

Since we know f_n by assumption, the quantity above being maximized is a known function of the single variable y. Denoting the maximizing y value as $y_{n+1}(c)$, we have

$$\begin{array}{lll} \text{Optimal} & \text{value} & f_{n+1}(c) = y_{n+1}(c) f_n(c - y_{n+1}(c)) \qquad (1) \\ \text{Optimal} & \text{strategy} & y_{n+1}(c) \text{ and optimal n-part strategy for } c - y_{n+1}(c) \qquad (2) \end{array}$$

The solution for $n = 2$ was given earlier. For $n = 3$ we get that

$$f_3(c) = \max_{0 \leq y \leq c} y(c - y)^2/4 = \frac{c}{3} \times f_2(2c/3) = \frac{c^3}{3^3},$$

as $h(y) = y(c-y)^2$ gives $h'(y) = (c-y)^2 - 2y(c-y) = 0$. This equation has two roots and the root $y = c$ corresponds to a global minimum of h on $[0, c]$ and so the only remaining root $c - y - 2y = 0$, i.e. $y = c/3$, corresponds to a global maximum of h on$[0, c]$ (the other interval end point $y = 0$ corresponds also to a global minimum of h on $[0, c]$).

Thus for $n = 3$, it holds that

$$\begin{array}{lll} \text{Optimal} & \text{value} & f_3(c) = c^3/3^3 \\ \text{Optimal} & \text{policy} & (c/3, c/3, c/3) \end{array}$$

The cases $n = 2$ and $n = 3$ as a hint, we conjecture that for general $n = 2, 3, 4, \ldots$, it holds that

$$\begin{array}{lll} \text{Optimal} & \text{policy} & (c/n, c/n, \ldots, c/n) \\ \text{Optimal} & \text{value} & f_n(c) = (c/n)^n \end{array}$$

We establish this result by induction. Assume that the optimal policy is as claimed for n. Then from (1)

$$f_{n+1}(c) = \max_{0\le y\le c} y\left(\frac{c-y}{n}\right)^n.$$

Consider the auxiliary function $h(y) = y(c-y)^n$ on $[0, c]$. Now $h'(y) = (c-y)^n - ny(c-y)^{n-1}$ and so

$$h'(y) = (c-y-ny)(c-y)^{n-1} = (c-(n+1)y)(c-y)^{n-1}.$$

The equation $h'(y) = 0$ has the roots $y = c/(n+1)$ and $y = c$ (the latter is a multiple root). The root $y = c$ and the other end point $y = 0$ of the interval $[0, c]$ correspond to a global minimum of h on $[0, c]$. The root $y = c/(n+1)$ corresponds clearly to a global maximum of h on $[0, c]$. Hence $y_{n+1}(c) = c/(n+1)$ is the optimal value of y.

Optimal	policy	$(c/(n+1), c/(n+1), \ldots, c/(n+1))$
Optimal	value	$f_{n+1}(c) = [c/(n+1)]^{n+1}$

So the result is valid for $n+1$ if it is valid for n. As we know that the result is valid for $n = 2$, it then follows that it is valid for all $n = 2, 3, 4, \ldots$. This completes the induction proof.

Consider a function $f(u_1, u_2)$ of two real variables u_1 and u_2. Let us assume that the minimum of f is obtained at $u_1 = u_1^o$, $u_2 = u_2^o$, i.e.

$$\min_{u_1,u_2} f(u_1, u_2) = f(u_1^o, u_2^o).$$

Then

$$\min_{u_1}\{\min_{u_2} f(u_1, u_2)\} \le \min_{u_1} f(u_1, u_2^o) \le f(u_1^o, u_2^o).$$

Hence it holds that

$$\min_{u_1,u_2} f(u_1, u_2) = \min_{u_1}\{\min_{u_2} f(u_1, u_2)\}.$$

Note that $\min_{u_2} f(u_1, u_2)$ is a function of u_1 only, so that we can write

$$\min_{u_1,u_2} f(u_1, u_2) = \min_{u_1} h(u_1), \tag{3}$$

where $h(u_1) = \min_{u_2} f(u_1, u_2)$.

Now consider the minimization of

$$f(x_1, u_1, x_2, u_2) = f_1(x_1, u_1) + f_2(x_2, u_2)$$

with respect to the decision variables u_1 and u_2 (u_1 and u_2 being the decision variables of the first stage and the second stage, respectively, of a two-stage decision problem). Here the state variables x_1 and x_2 are such that the state x_2 of the second stage depends on u_1 and x_1, i.e.

$$x_2 = g(x_1, u_1),$$

where g is a given function. By (3) the minimum of f, with respect to u_1 and u_2, under the above constraint can be written as

$$\min_{u_1,u_2} f(x_1, u_1, x_2, u_2) = \min_{u_1,u_2} (f_1(x_1, u_1) + f_2(x_2, u_2)) =$$
$$\min_{u_1} (f_1(x_1, u_1) + \min_{u_2} f_2(g(x_1, u_1), u_2)) =$$
$$\min_{u_1} [f_1(x_1, u_1) + V(g(x_1, u_1))],$$

where we have introduced the notation

$$V(x_2) = \min_{u_2} f_2(x_2, u_2).$$

More generally, we consider a sequential decision process consisting of N stages with the decision variables u_1, u_2, ..., u_N, and the loss function

$$f(x_1, u_1, \ldots, x_N, u_N) = f_1(x_1, u_1) + \ldots + f_N(x_N, u_N),$$

where the state variables x_1,. . .,x_N are related according to

$$x_{k+1} = g_k(x_k, u_k), \quad k = 1, \ldots, N-1.$$

Here g_k, $k = 1, \ldots, N-1$, are given functions.

We obtain using (3) repeatedly

$$\begin{aligned}
&\min_{u_1,\ldots,u_N} f(x_1, u_1, \ldots, x_N, u_N) = \\
&\min_{u_1,\ldots,u_{N-1}} [f_1(x_1, u_1) + \ldots + f_{N-1}(x_{N-1}, u_{N-1}) + \min_{u_N} f_N(g_{N-1}(x_{N-1}, u_{N-1}), u_N)] = \\
&\min_{u_1}[f_1(x_1, u_1) + \min_{u_2}[f_2(g_1(x_1, u_1), u_2) + \min_{u_3}[f_3(g_2(x_2, u_2), u_3) + \ldots + \\
&\min_{u_{N-1}}[f_{N-1}(g_{N-2}(x_{N-2}, u_{N-2}), u_{N-1}) + \min_{u_N} f_N(g_{N-1}(x_{N-1}, u_{N-1}), u_N)] \ldots], \qquad (4)
\end{aligned}$$

where x_{k-1} does not depend on u_k, $k = 2, \ldots, N$. The above optimization problems are to be performed starting from the decision variable u_N and then proceeding recursively backwards to u_1. That is, the above procedure can be expressed recursively as

$$\min_{u_1,u_2,\ldots,u_N} f(x_1, u_1, \ldots, x_N, u_N) = V_1(x_1),$$

where $V_1(x_1)$ is given by the recursive functional equation

$$V_k(x_k) = \min_{u_k}[f_k(x_k, u_k) + V_{k+1}(g_k(x_k, u_k))], \quad k = N-1, \ldots, 1 \qquad (5)$$

with the initial condition

$$V_N(x_N) = \min_{u_N} f_N(x_N, u_N).$$

Remark 2.1 Comparing (4) and (5) it is seen that $V_k(x_k)$ is the minimum of the contribution to the total loss function f from the stages k, ...,N, as a function of the state x_k at stage k, i.e.

$$V_k(x_k) = \min_{u_k,\ldots,u_N} [f_k(x_k, u_k) + f_{k+1}(x_{k+1}, u_{k+1}) + \ldots + f_N(x_N, u_N)]$$

subject to the state constraints

$$x_{k+1} = g_k(x_k, u_k), \quad k = 1, \ldots, N-1.$$

The functional equation (5) (with the associated initial condition) is called the **Bellman equation** – it is the basis for **dynamic programming** in which the sequential optimization problem $\min_{u_1,\ldots,u_N} f(x_1, u_1, \ldots, x_N, u_N)$ subject to the state constraints $x_{k+1} = g_k(x_k, u_k)$, $k = 1, \ldots, N-1$, is solved as a sequence of smaller subproblems of the form (5). These subproblems also define the optimal strategy $u_k^* = u_k^*(x_k)$, $k = N, N-1, \ldots, 1$.

2.2 Stochastic Sequential Optimization Problems

Whilst deterministic sequential optimization problems can often be solved by for example direct optimization, stochastic problems often can only be solved with dynamic programming. Why?

The reason is that the information that is available at the various stages when selecting u_1, ..., u_N (i.e. when determining the decision policy or the control strategy) is different at each stage: when u_{k+1} is determined, there are in general new measurements of stochastic variables available and one stage earlier only the conditional distributions of these variables were available when selecting u_k.

Before proceding we need some auxiliary results.

Auxiliary Results

Let X and Y be stochastic variables, and let u be a decision variable (control signal) which is to be chosen so that the loss function

$$E[\ell(X, Y, u)]$$

is minimized. (Here E denotes the expectation (or mean value) over the random variables X and Y, and $\ell(\cdot,\cdot,\cdot)$ is a given function.) The decision variable u is allowed to be a function of y only (an observation of the stochastic variable Y). That is, the minimization problem can be formulated as

$$\min_{u=u(Y)} E[\ell(X, Y, u)].$$

It is convenient to introduce the simplified notation $E[\cdot \mid y]$ to denote the conditional expectation $E[\cdot \mid Y = y]$.

Result A

Assume that the function $f(y,u) = E[\ell(X,y,u) \mid y]$ for every y has a unique minimum with respect to u, and let this minimum be achieved for $u^o(y)$. Then

$$\min_{u(Y)} E[\ell(X,Y,u(Y))] = E[\ell(X,Y,u^o(Y))] = E_Y[\min_u E[\ell(X,y,u) \mid y]]. \tag{6}$$

This we see as follows. We should first note that $f(y,u)$ is a function of y and u only. It follows that the explicit dependence of u on y can be left out in the minimization, i.e.

$$\min_{u(y)} E[\ell(X,y,u) \mid y] = \min_u E[\ell(X,y,u) \mid y].$$

For every $u = u(y)$ we have that

$$f(y,u) \geq f(y,u^o(y)) = \min_u f(y,u),$$

so that we have for every $u = u(y)$ the useful relationships

$$\begin{aligned}
E[\ell(X,Y,u)] &= E_Y[E[\ell(X,Y,u) \mid y]] \\
&= E_Y[f(Y,u)] \\
&\geq E_Y[f(Y,u^o(Y))], \qquad (7)\\
&= E_Y[E[\ell(X,y,u^o(y)) \mid y]] \\
&= E[\ell(X,Y,u^o(Y))] \qquad (8)\\
&= E_Y[\min_u f(y,u)], \\
&= E_Y[\min_u E[\ell(X,y,u) \mid y]]. \qquad (9)
\end{aligned}$$

Here the first equality follows by the law of total probability for expectations, see Part I of these lecture notes and the second by the definition of $f(y,u)$. The inequality (7) follows by the definition of $u^o(y)$. The equality (8) follows by the total law of probability for expectations and the quantity on the right hand side of the equality after (8) is equal to the right hand side of the inequality (7). Equality (9) follows by the definition of $f(y,u)$.

Hence by (8) and (9)

$$\min_{u(Y)} E[\ell(X,Y,u(Y))] \geq E[\ell(X,Y,u^o(Y))] = E_Y[\min_u E[\ell(X,y,u) \mid y]].$$

On the other hand it clearly holds that

$$E[\ell(X,Y,u^o(Y))] \geq \min_{u(Y)} E[\ell(X,Y,u(Y))]$$

(actually equality holds here) and thus

$$\min_{u(Y)} E[\ell(X,Y,u(Y))] = E_Y[\min_u E[\ell(X,y,u) \mid y]].$$

Thus we have proved the validity of (6).

When u is allowed to be a function of x (an observation of the stochastic variable X) as well, we obtain analogously to Result A:

Result B

Assume that the function $\ell(x,y,u)$ has a unique minimum for every (x,y) with respect to u, and let this minimum be attained for $u^o(x,y)$. Then

$$\min_{u(X,Y)} E[\ell(X,Y,u(X,Y))] = E[\ell(X,Y,u^o(X,Y))] = E[\min_{u=u(X,Y)} \ell(X,Y,u)]. \tag{10}$$

Actually, we can compute as in Result A to get

$$\min_{u(X,Y)} E[\ell(X,Y,u(X,Y))] = E_{X,Y}[\min_u \ell(x,y,u) \mid x,y] = E[\min_{u=u(X,Y)} \ell(X,Y,u)].$$

Let us now return back to stochastic sequential optimization problems. Consider first the problem of minimization of the function $E[\ell(X_1,X_2,u_1,u_2)]$, where X_1 and X_2 are stochastic variables and

$$\ell(X_1,X_2,u_1,u_2) = \ell_1(X_1,u_1) + \ell_2(X_2,u_2).$$

The vector x_1 (an observation of the stochastic variable X_1) represents the information available at the first stage, and u_1 is allowed to be a function of this information only. The vector x_2 (an observation of the stochastic variable X_2) represents the information available at the second stage, and u_2 is allowed to depend on this information only.

Note that in contrast to the deterministic case, where x_2 is available at stage 2 for u_2 via the relationship $x_2 = g_1(x_1,u_1)$, in the stochastic case only the conditional distribution of X_2 given x_1 and u_1 is available at stage 2 for u_2, for example via a relationship of the form $x_2 = g(x_1,u_1,w)$, where w is a stochastic variable.

We get

$$\begin{aligned}\min\nolimits_{u_1(X_1),u_2(X_2)} E[\ell(X_1,X_2,u_1,u_2)] = \min\nolimits_{u_1(X_1)}\{\min\nolimits_{u(X_2)} E[\ell(X_1,X_2,u_1,u_2)]\} =\\ \min\nolimits_{u(X_1)}\{E[\ell_1(X_1,u_1)] + \min\nolimits_{u_2(X_2)} E[\ell_2(X_2,u_2)]\}\end{aligned}$$

since u_2 does not affect $\ell_1(X_1,u_1)$. Here we have used the notation $u_1 = u_1(X_1)$ and $u_2 = u_2(X_2)$. We use now result B (10) to get finally that

$$\min_{u_1(X_1),u_2(X_2)} E[\ell(X_1,X_2,u_1,u_2)] = \min_{u_1(X_1)}\{E[\ell_1(X_1,u_1)] + E[\min_{u_2} \ell_2(X_2,u_2)]\}.$$

Introduce the function

$$V(x_2) = \min_{u_2} \ell_2(x_2, u_2).$$

Then we can write

$$\begin{aligned}\min_{u_1(X_1),u_2(X_2)} E[\ell(X_1, X_2, u_1, u_2)] = \min_{u_1(X_1)}\{E[\ell_1(X_1, u_1) + V(X_2)]\} = \\ E_{X_1}\{\min_{u_1}[\ell_1(x_1, u_1) + E[V(X_2) \mid x_1]]\} = \\ E_{X_1}\{\min_{u_1}[\ell_1(x_1, u_1) + E[V(X_2) \mid x_1, u_1]]\} \end{aligned} \tag{11}$$

To derive the second equality we have used (10) and the fact that $E[\ell_1(x_1, u_1) \mid x_1] = \ell_1(x_1, u_1)$. Note that we have wanted to stress in (11) that the latter term is a function of x_1 as well as of u_1 by writing $E[V(X_2) \mid x_1, u_1]$.

The result (11) can be generalized to the minimization of

$$E[\ell_1(X_1, u_1) + \ell_2(X_2, u_2) + \ldots + \ell_N(X_N, u_N)], \tag{12}$$

where x_k (an observation of the stochastic variable X_k) is the available information for u_k at stage k, at which only the conditional distributions of X_{k+1}, …, X_N, are known, for example via relationships of the form $x_{i+1} = g_i(x_i, u_i, w_i)$, $i = k, \ldots, N-1$, where w_i are stochastic variables.

We obtain

$$\begin{aligned}\min_{u_1(X_1),\ldots,u_N(X_N)} E[\ell_1(X_1, u_1) + \ldots + \ell_{N-1}(X_{N-1}, u_{N-1}) + \ell_N(X_N, u_N)] = \\ \min_{u_1(X_1),\ldots,u_{N-1}(X_{N-1})}\{E[\ell_1(X_1, u_1) + \ldots + \ell_{N-1}(X_{N-1}, u_{N-1})] + E[\min_{u_N} \ell_N(X_N, u_N)]\} \\ = \min_{u_1(X_1),\ldots,u_{N-1}(X_{N-1})}\{E[\ell_1(X_1, u_1) + \ldots + \ell_{N-1}(X_{N-1}, u_{N-1}) + V_N(X_N)]\},\end{aligned}$$

where

$$V_N(x_N) = \min_{u_N} \ell_N(x_N, u_N).$$

Here we have used the previously derived results for the two-stage situation.

Repeating the above steps with stage $N-1$ gives

$$\begin{aligned}\min_{u_1(X_1),\ldots,u_N(X_N)} E[\ell_1(X_1, u_1) + \ldots + \ell_{N-1}(X_{N-1}, u_{N-1}) + \ell_N(X_N, u_N)] = \\ \min_{u_1(X_1),\ldots,u_{N-2}(X_{N-2})}\{E[\ell_1(X_1, u_1) + \ldots + \ell_{N-2}(X_{N-2}, u_{N-2})] \\ + \min_{u_{N-1}(X_{N-1})} E[\ell_{N-1}(X_{N-1}, u_{N-1}) + V_N(X_N)]\}\end{aligned} \tag{13}$$

In analogy with (11) we get

$$\begin{aligned}\min_{u_{N-1}(X_{N-1})} E[\ell_{N-1}(X_{N-1}, u_{N-1}) + +V_N(X_N)] = \\ E\{\min_{u_{N-1}}[\ell_{N-1}(x_{N-1}, u_{N-1}) + E[V_N(X_N) \mid x_{N-1}, u_{N-1}]]\} = \\ E[V_{N-1}(X_{N-1})],\end{aligned} \tag{14}$$

where we have introduced the notation

$$V_{N-1}(X_{N-1}) = \min_{u_{N-1}}[\ell_{N-1}(x_{N-1}, u_{N-1}) + E[V_N(X_N) \mid x_{N-1}, u_{N-1}]].$$

Inserting (14) into (13) gives

$$\min_{u_1(X_1),\ldots,u_N(X_N)} E[\ell_1(X_1, u_1) + \ldots + \ell_{N-1}(X_{N-1}, u_{N-1}) + \ell_N(X_N, u_N)] =$$
$$\min_{u_1(X_1),\ldots,u_{N-2}(X_{N-2})}\{E[\ell_1(X_1, u_1) + \ldots + \ell_{N-2}(X_{N-2}, u_{N-2})] + V_{N-1}(X_{N-1})]\}$$

Repeating this procedure for u_{N-2}, …, u_1, gives

$$\min_{u_1(X_1),\ldots,u_N(X_N)} E[\ell_1(X_1, u_1) + \ldots + \ell_{N-1}(X_{N-1}, u_{N-1}) + \ell_N(X_N, u_N)] = E[V_1(X_1)], \quad (15)$$

where the function $V_1(x_1)$ is obtained using functions $V_k(x_k)$ which are determined recursively from the functional equation

$$V_k(x_k) = \min\{\ell_k(x_k, u_k) + E[V_{k+1}(X_{k+1}) \mid x_k, u_k]\},\ k = N-1, \ldots, 1 \quad (16)$$

with the initial condition

$$V_N(x_N) = \min_{u_N} \ell_N(x_N, u_N). \quad (17)$$

These are the Bellman equations for the stochastic dynamic programming problem.

Note that from the construction of the functions $V_k(x_k)$, it follows that $V_k(x_k)$ is the expected minimum of the contribution of the loss function (12) from stages k, …, N, given the information x_k available at stage k, i.e.

$$V_k(x_k) = \min_{u_k(X_k),\ldots,u_N(X_N)} E[\ell_k(X_k, u_k) + \ldots + \ell_N(X_N, u_N) \mid x_k, u_k],\ k = N, N-1, \ldots, 1.$$

2.3 Incomplete State Information

Let us now generalize the solution (15)-(16) to the case when x_k is not available at stage k, and thus u_k is restricted to be a function of the information y_k (an observation of the stochastic variable Y_k). Hence only the conditional distribution of X_k given y_k is known.

For example, information about X_k may be given via the relationships

$$\begin{aligned} x_{k+1} &= g_k(x_k, u_k, w_k) \\ y_k &= h_k(x_k, v_k), \end{aligned}$$

where w_k and v_k are stochastic variables and $k = 1, \ldots, N-1$.

First consider a two-stage problem with the loss function

$$E[\ell(X_1, Y_1, X_2, Y_2, u_1, u_2)] = E[\ell_1(X_1, Y_1, u_1) + \ell_2(X_2, Y_2, u_2)]$$

which is to be minimized with respect to $u_1 = u_1(y_1)$ (stage 1) and $u_2 = u_2(y_2)$ (stage 2). At stage 1, y_1 and the conditional distributions of X_1 and X_2 given y_1, u_1 are known. For example, we may have $x_2 = g(x_1, u_1, w)$, $y_1 = h_1(x_1, v_1)$ (and also $y_2 = h_2(x_2, v_2)$). We obtain

$$\min\nolimits_{u_1(Y_1),u_2(Y_2)} E[\ell(X_1, Y_1, X_2, Y_2, u_1, u_2)] =$$
$$\min\nolimits_{u_1(Y_1)}\{\min\nolimits_{u_2(Y_2)} E[\ell(X_1, Y_1, X_2, Y_2, u_1, u_2)]\} =$$
$$\min\nolimits_{u_1(Y_1)}\{E[\ell_1(X_1, Y_1, u_1)] + \min\nolimits_{u_2(Y_2)} E[\ell_2(X_2, Y_2, u_2)]\} =$$
$$\min\nolimits_{u_1(Y_1)}\{E[\ell_1(X_1, Y_1, u_1)] + E[\min\nolimits_{u_2} E[\ell_2(X_2, y_2, u_2) \mid y_2]]\},$$

where the second equality follows since u_2 does not affect $\ell_1(X_1, Y_1, u_1)$ and the last equality follows from (10) (see Result B). (Here in the last expression, the outer expectation operation E is with respect to the stochastic variable Y_2.)

Introduce the function

$$V(y_2) = \min_{u_2} E[\ell_2(X_2, y_2, u_2) \mid y_2].$$

(Note that here the expectation is with respect to (w.r.t.) the distribution of X_2, and NOT w.r.t. the distribution of Y_2.) Then

$$\min\nolimits_{u_1(Y_1),u_2(Y_2)} E[\ell(X_1, Y_1, X_2, Y_2, u_1, u_2)] =$$
$$\min\nolimits_{u_1(Y_1)} E[\ell_1(X_1, Y_1, u_1) + V(Y_2)] =$$
$$E\{\min\nolimits_{u_1} E[\ell_1(X_1, y_1, u_1) + V(Y_2) \mid y_1]\} =$$
$$E\{\min\nolimits_{u_1} E[\ell_1(X_1, y_1, u_1) + V(Y_2) \mid y_1, u_1]\}$$

where we have wanted to stress (with the notation used) that Y_2 (and hence $V(Y_2)$) is a function of u_1, too. (For example, as in $x_2 = g(x_1, u_1, w)$, $y_2 = h_2(x_2, v_2)$.)

This procedure generalizes to the multistage loss function

$$E[\ell_1(X_1, Y_1, u_1) + \ldots + \ell_N(X_N, Y_N, u_N)].$$

In analogy with the derivation of (15)-(17) we have (sic!)

$$\min_{u_1(Y_1),\ldots,u_N(Y_N)} E[\ell_1(X_1, Y_1, u_1) + \ldots + \ell_N(X_N, Y_N, u_N)] = E[V_1(Y_1)], \tag{18}$$

where the function $V_1(y_1)$ is obtained from the functions $V_k(y_k)$, which are given by the recursive equation

$$V_k(y_k) = \min_{u_k} E[\ell_k(X_k, y_k, u_k) + V_{k+1}(Y_{k+1}) \mid y_k, u_k],\ k = N-1, \ldots, 1 \tag{19}$$

with the initial condition

$$V_N(y_N) = \min_{u_N} E[\ell_N(X_N, y_N, u_N) \mid y_N]. \tag{20}$$

Note also that

$$V_k(y_k) = \min_{u_k(Y_k),\ldots,u_N(Y_N)} E[\ell_k(X_k, y_k, u_k) + \ldots + \ell_N(X_N, Y_N, u_N) \mid y_k, u_k].$$

(See also the end of the previous subsection.)

3 The Linear Quadratic Gaussian Control Problem

In this section we shall study the Linear Quadratic Gaussian (LQG) control problem, whose solution is based on stochastic dynamic programming (SDP). The solution of the LQG control problem is one of the most celebrated results in the control and systems field. (There are actually several LQG control problems depending on the information available for computing the control law.)

3.1 Statement of the LQG Control Problem

We shall consider linear state space systems corrupted with normally distributed disturbances as in Part I of these lectures. For convenience we shall reproduce the state space equations below.

Thus we consider the state space system

$$x(t+1) = Ax(t) + Bu(t) + w(t) \tag{21}$$

$$y(t) = Cx(t) + v(t) \tag{22}$$

where $\{w(t)\}$ and $\{v(t)\}$ are sequences of independent *normally distributed* vectors (stochastic variables) with zero mean values and the covariance matrices

$$\begin{aligned} E[w(t)w(s)^T] &= R_1\delta_{t,s} \\ E[v(t)v(s)^T] &= R_2\delta_{t,s} \\ E[w(t)v(s)^T] &= 0 \end{aligned} \tag{23}$$

where $\delta_{t,s} = 1$ for $t = s$ and $\delta_{t,s} = 0$ for $t \neq s$ ($\delta_{t,s}$ is the so-called Kronecker delta). It is assumed that the initial state $x(t_0)$ is normally distributed with mean value m and covariance matrix R_0, and that the initial state is independent of $\{w(t)\}$ and $\{v(t)\}$.

As control criterion we take the scalar quadratic loss function

$$J_N \equiv E[x(N)^T Q_0 x(N) + \sum_{t=t_0}^{N-1} \{x(t)^T Q_1 x(t) + u(t)^T Q_2 u(t)\}], \tag{24}$$

where $N > t_0$ and Q_0, Q_1 and Q_2 are symmetric and positive definite or (positive) semidefinite matrices (of appropriate dimensions).

The control problem can be stated as follows.

Problem 1 Find an *admissible* control strategy for the system (21), (22), (23) for which the loss function (24) is minimized.

An admissible control strategy is such that $u(t)$ is a function of the information available at time instant t only.

We will consider two cases:

- Complete state information, that is, the whole state vector $x(t)$ is assumed known at time instant t.
- Incomplete state information, that is, either the information $I_1(t) = \{y(t-1), u(t-1), y(t-2), u(t-2), \ldots\}$ or the information $I_0(t) = \{y(t), y(t-1), u(t-1), y(t-2), u(t-2), \ldots\}$ is assumed to be available to compute $u(t)$. Obviously we should use $I_0(t)$ rather than $I_1(t)$ whenever possible when computing $u(t)$ (in any particular application).

3.2 Solution of the LQG Problem: Complete State Information

We solve here Problem 1 when $u(t)$ is a function of the state vector $x(t)$.

The loss function (24) can be written as

$$J_N = E[\ell(x(N)) + \sum_{t=t_0}^{N-1} \ell_t(x(t), u(t))],$$

where

$$\begin{aligned} \ell_t(x(t), u(t)) &= x(t)^T Q_1 x(t) + u(t)^T Q_2 u(t),\ t = t_0, \ldots, N-1 \\ \ell_N(x(N)) &= x(N)^T Q_0 x(N). \end{aligned}$$

The control problem has now been written in a form suitable for stochastic dynamic programming (SDP) with complete state information. That is, we shall use the recursive SDP procedure (15),(16), (17). This gives the solution to Problem 1 with complete (full) state information in the form

$$\min_{u(t_0),\ldots,u(N-1)} J_N = E[V_{t_0}(x(t_0))], \tag{25}$$

where (for $t = N-1, \ldots, t_0$)

$$V_t(x(t)) = \min_{u(t)} \{x(t)^T Q_1 x(t) + u(t)^T Q_2 u(t) + E[V_{t+1}(x(t+1)) \mid x(t), u(t)]\} \tag{26}$$

with the initial condition

$$V_N(x(N)) = \ell_N(x(N)) = x(N)^T Q_0 x(N). \tag{27}$$

We shall next solve the functional Bellman equation (26). For this we use an induction argument showing that the solution of (26) has the form

$$V_t(x(t)) = x(t)^T S(t) x(t) + s(t), \tag{28}$$

where $S(t)$ is a symmetric, positive definite or semidefinite matrix.

We start by noting that clearly (28) holds for $t = N$ with $S(N) = Q_0$ and $s(N) = 0$, compare (27).

We shall next show that if (28) holds for $t+1$, then it will also hold for t. This then implies that (28) holds for $t = N, N-1, \ldots, t_0$.

Thus we put

$$V_{t+1}(x(t+1)) = x(t+1)^T S(t+1) x(t+1) + s(t+1).$$

By (26) we need to evaluate the conditional mean $E[V_{t+1}(x(t+1)) \mid x(t), u(t)]$. Recall (21)

$$x(t+1) = Ax(t) + Bu(t) + w(t),$$

so that $x(t+1)$ given $x(t)$ and $u(t)$, is normally distributed with mean value

$$E[x(t+1) \mid x(t), u(t)] = Ax(t) + Bu(t)$$

and the covariance matrix

$$E[\{x(t+1) - (Ax(t) - Bu(t))\}\{x(t+1) - (Ax(t) + Bu(t))\}^T \mid x(t), u(t)] = E[w(t)w(t)^T] = R_1.$$

To proceed we need the following auxiliary result.

<u>Result:</u> Let x be normally distributed with mean value m and covariance matrix R, and let S be a given (square) matrix (of the same dimensions as R). Then

$$E[x^T S x] = m^T S m + \operatorname{tr} SR \tag{29}$$

(Recall from part I that $\operatorname{tr}(\cdot)$ denotes the trace of a square matrix, i.e. the sum of the diagonal elements of the square matrix.) This we see as follows. We compute

$$\begin{aligned} E[x^T S x] &= E[(x-m)^T S (x-m)] + E[m^T S x] + E[x^T S m] - E[m^T S m] \\ &= E[(x-m)^T S (x-m)] + m^T S m, \end{aligned}$$

as $E[x] = m$. Thus

$$\begin{aligned} E[x^T S x] &= E[\operatorname{tr}((x-m)^T S (x-m))] + m^T S m \\ &= E[\operatorname{tr}(S(x-m)(x-m)^T)] + m^T S m \\ &= \operatorname{tr}(SE[(x-m)(x-m)^T]) + m^T S m \\ &= \operatorname{tr} SR + m^T S m. \end{aligned}$$

(Here we have used the equality $\operatorname{tr} UV = \operatorname{tr} VU$ valid for any dimension compatible matrices U and V.)

The result (29) now gives that

$$E[V_{t+1}(x(t+1)) \mid x(t), u(t)] = E[x(t+1)^T S(t+1)x(t+1) + s(t+1) \mid x(t), u(t)] =$$
$$(Ax(t) + Bu(t))^T S(t+1)(A(x(t) + Bu(t)) + \operatorname{tr} R_1 S(t+1) + s(t+1).$$

Inserting this into the Bellman equation (26) gives

$$\begin{gathered} V_t(x(t)) = \min_{u(t)}\{x(t)^T Q_1 x(t) + u(t)^T Q_2 u(t) + \\ (Ax(t) + Bu(t))^T S(t+1)(A(x(t) + Bu(t)) + \operatorname{tr} R_1 S(t+1) + s(t+1)\} = \\ \min_{u(t)}\{u(t)^T (B^T S(t+1)B + Q_2)u(t) + u(t)^T B^T S(t+1)Ax(t) + \\ x(t)^T A^T S(t+1)Bu(t)\} + \\ x(t)^T (A^T S(t+1)A + Q_1)x(t) + \operatorname{tr} R_1 S(t+1) + s(t+1), \end{gathered} \tag{30}$$

where we have moved all terms that do not depend on $u(t)$ outside the minimization operation with respect to $u(t)$. Completing the squares according to the identity (with $M^T = M$)

$$u^T M u + u^T N x + x^T N^T u =$$
$$(Mu + Nx)^T M^{-1}(Mu + Nx) - x^T N^T M^{-1} N x$$

gives

$$\begin{gathered} V_t(x(t)) = \min_{u(t)}\{[(B^T S(t+1)B + Q_2)u(t) + B^T S(t+1)Ax(t)]^T (B^T S(t+1) + Q_2)^{-1} \times \\ [(B^T S(t+1)B + Q_2)u(t) + B^T S(t+1)Ax(t)] \\ -x(t)^T A^T S(t+1)B(B^T S(t+1)B + Q_2)^{-1} B^T S(t+1)Ax(t) + \\ x(t)^T (A^T S(t+1)A + Q_1)x(t) + \operatorname{tr} R_1 S(t+1) + s(t+1)\}, \end{gathered}$$

where we have assumed that the square matrix $(B^T S(t+1)B + Q_2)$ is invertible.

The solution of this minimization problem is now seen to be obtained for the $u(t)$ satisfying

$$(B^T S(t+1)B + Q_2)u(t) + B^T S(t+1)Ax(t) = 0.$$

(The first term in the expression being minimized is the only term that depends on $u(t)$ and this first term is nonnegative for any $u(t)$, so the best we can do is to make this term equal to zero.) This gives the optimal control strategy as

$$u(t) = -(B^T S(t+1)B + Q_2)^{-1} B^T S(t+1)Ax(t) \tag{31}$$

(Thus the optimal control strategy is a linear control law feeding back the state $x(t)$.) This gives

$$V_t(x(t)) = -x(t)^T A^T S(t+1)B(B^T S(t+1)B + Q_2)^{-1} B^T S(t+1)Ax(t) +$$

$$x(t)^T(A^TS(t+1)A+Q_1)x(t)+\operatorname{tr}R_1S(t+1)+s(t+1)=$$
$$x(t)^T[A^TS(t+1)A-A^TS(t+1)B(B^TS(t+1)B+Q_2)^{-1}B^TS(t+1)A+Q_1]x(t)+$$
$$\operatorname{tr}R_1S(t+1)+s(t+1).$$

Thus $V_t(x(t))$ is of the form (28) with

$$\begin{aligned}S(t) &= A^TS(t+1)A-A^TS(t+1)B(B^TS(t+1)B+Q_2)^{-1}B^TS(t+1)A+Q_1\\ s(t) &= s(t+1)+\operatorname{tr}R_1S(t+1)\end{aligned}$$

We still need to verify that $S(t)$ is at least positive semidefinite (it is symmetric by the above expression). It is convenient to introduce the feedback gain matrix $L(t)$ as

$$L(t)=(B^TS(t+1)B+Q_2)^{-1}B^TS(t+1)A.$$

(Thus the optimal control strategy is $u(t)=-L(t)x(t)$.) Inserting this into the obtained expression for $S(t)$, we can write

$$\begin{aligned}S(t) &= (A-BL(t))^TS(t+1)(A-BL(t))+\\ &\quad L(t)^TB^TS(t+1)A-L(t)^TB^TS(t+1)BL(t)+Q_1\\ &= (A-BL(t))^TS(t+1)(A-BL(t))+L(t)^TB^TS(t+1)A\\ &\quad -L(t)^T(B^TS(t+1)B+Q_2)L(t)+L(t)^TQ_2L(t)+Q_1\\ &= (A-BL(t))^TS(t+1)(A-BL(t))+L(t)^TB^TS(t+1)A-L(t)^TB^TS(t+1)A+\\ &\quad L(t)^TQ_2L(t)+Q_1\\ &= (A-BL(t))^TS(t+1)(A-BL(t))+L(t)^TQ_2L(t)+Q_1.\end{aligned}$$

Hence

$$z^TS(t)z=d^TS(t+1)d+h^TQ_2h+z^TQ_1z$$

with $d=(A-BL(t))z$ and $h=L(t)z$. It follows that $z^TS(t)z\geq 0$ for any vector z as $S(t+1)$, Q_2 and Q_1 are (symmetric) positive semidefinite matrices by assumption. Thus $S(t)$ is (symmetric) positive semidefinite (at least; it may be positive definite). This completes the induction proof of the claimed solution (28) to the LQG control problem with complete state information.

Solution to Problem 1 with Complete State Information – Summary

Let an admissible control strategy be such that $u(t)$ is a function of $x(t)$. The loss function (24) is then minimized by the control strategy

$$u(t)=-L(t)x(t), \tag{32}$$

where

$$L(t)=(B^TS(t+1)B+Q_2)^{-1}B^TS(t+1)A \tag{33}$$

and $S(t)$ is given by the recursive equation

$$S(t) = A^T S(t+1)A - A^T S(t+1)B(B^T S(t+1)B + Q_2)^{-1}B^T S(t+1)A + Q_1 \qquad (34)$$

for $t = N-1, \dots, t_0$. The initial condition for (34) is

$$S(N) = Q_0.$$

If the initial state $x(t_0)$ has mean value m and covariance matrix R_0, then the minimum of the loss function (24), or equivalently the minimum value in (25), is given by

$$\begin{aligned}
\min_{u(t_0),\dots,u(N-1)} J_N &= E[V_{t_0}(x(t_0))] \\
&= E[x(t_0)^T S(t_0)x(t_0) + s(t_0)] \\
&= m^T S(t_0)m + \operatorname{tr} S(t_0)R_0 + s(t_0) \\
&= m^T S(t_0)m + \operatorname{tr} S(t_0)R_0 + \sum_{t=t_0}^{N-1} \operatorname{tr} S(t+1)R_1,
\end{aligned}$$

where the last equality follows by the earlier obtained recursive expression for $s(t)$ as $s(N) = 0$.

Remark 3.1 Note that the matrix equation (34) is called a Riccati equation. It is dual to the Riccati equation for the prediction error covariance matrix in Kalman filtering, see part I of these lecture notes.

Remark 3.2 As $N - t \to \infty$ (for example when $N \to \infty$ or $t_0 \to -\infty$), (34) converges under certain (rather mild) conditions to the stationary (or algebraic) Riccati equation

$$S = A^T SA - A^T SB(B^T SB + Q_2)^{-1}B^T SA + Q_1$$

or

$$S = (A - BL)^T S(A - BL) + L^T Q_2 L + Q_1,$$

where $S = \lim_{t\to-\infty} S(t)$ and

$$L = \lim_{t\to-\infty} L(t) = (B^T SB + Q_2)^{-1}B^T SA.$$

The *stationary control law*

$$u(t) = -Lx(t)$$

then minimizes the *stationary loss function*

$$J = \lim_{N\to\infty} E\left[\frac{1}{N}\sum_{t=0}^{N-1}\left(x(t)^T Q_1 x(t) + u(t)^T Q_2 u(t)\right)\right]$$

(that is, the average loss per step). The minimum value of the stationary loss is

$$\min_{u(t)} J = \lim_{N\to\infty} \min_{u(t)} \frac{1}{N} J_N = \lim_{N\to\infty} \frac{1}{N} \sum_{t=0}^{N-1} \operatorname{tr} S(t+1) R_1 = \operatorname{tr} S R_1.$$

(We have denoted $t_0 = 0$.)

Remark 3.3 Note that the auxiliary result (29) holds for any probability distribution with finite mean and covariance matrix. We therefore do not actually need the normality assumption for the (process) noise $\{w(t)\}$ for the result (32)-(34) to hold. It suffices that the disturbance $\{w(t)\}$ is white noise. ($x(t)$ must be independent of $w(t)$ – this is the essential assumption needed. Note that knowledge of $x(t)$, $x(t-1)$, ..., $u(t-1)$, $u(t-2)$, ..., allows one to determine $w(t-1)$, $w(t-2)$, ... : $w(t-1) = x(t) - Ax(t-1) - Bu(t-1)$; $w(t-2) = x(t-1) - Ax(t-2) - Bu(t-2)$; and so on. So if $\{w(t)\}$ would not be white noise then $x(t)$ would contain information about $w(t)$ –> we must assume $\{w(t)\}$ to be white noise – standard assumption for state space models!)

Remark 3.4 Note that the control strategy (32)-(34) does not depend on the covariance matrix R_1 (of $w(t)$). In fact the control law (32)-(34) is optimal also in the deterministic case, when $R_1 = 0$ and $R_0 = 0$. That is, it is optimal for *the deterministic initial value problem* defined by

$$x(t+1) = Ax(t) + Bu(t), \quad x(t_0) = m = \text{given},$$

with the quadratic loss function

$$J_N = x(N)^T Q_0 x(N) + \sum_{t=t_0}^{N-1} \left[x(t)^T Q_1 x(t) + u(t)^T Q_2 u(t) \right].$$

Therefore the same computer aided control system design (CACSD) software can be used to solve both stochastic and deterministic linear quadratic (LQ) control problems.

3.3 Solution of the LQG Problem: Incomplete State Information

We solve here Problem 1 when the state vector $x(t)$ need not be available for the computation of $u(t)$, and instead there are corrupted measurements of some linear combinations of the state vector at disposal. There are two such cases of interest to us here corresponding to the information $I_1(t)$ and $I_0(t)$, respectively (see the discussion after the statement of Problem 1).

Let us now solve the optimal control problem when the information

$$I_1(t) = [y(t_0)^T, u(t_0)^T, \ldots, y(t-1)^T, u(t-1)^T]^T$$

is available for the computation of $u(t)$, i.e. when $u(t)$ is a function of $I_1(t)$. (Note that in Part I of these lecture notes, we have denoted the information state $I_1(t)$ as V_{t-1}, but the latter notation is here reserved to a quantity in the Bellman equation.)

In analogy with (25),(26) and (27) we obtain, using (18),(19) and (20) (with the substitutions $k \to t$, $x_k \to x(t)$ and $y_k \to I_1(t)$):

$$\min_{u(t_0),\ldots,u(N-1)} J_N = E[V_{t_0}(I_1(t_0))], \tag{35}$$

where $V_{t_0}(\cdot)$ is obtained recursively according to (the Bellman equation) (for $t = N - 1, \ldots, t_0$)

$$V_t(I_1(t)) = \min_{u(t)} E[x^T Q_1 x(t) + u(t)^T Q_2 u(t) + V_{t+1}(I_1(t+1)) \mid I_1(t), u(t)] \tag{36}$$

with the initial condition

$$V_N(I_1(N)) = E[x(N)^T Q_0 x(N) \mid I_1(N)]. \tag{37}$$

The dimension of the information vector $I_1(t)$ depends on t. Now $I_1(t) = [I_1(t-1)^T, y(t)^T, u(t)^T]^T$, where $y(t)$ is given by (see (22))

$$y(t) = Cx(t) + v(t).$$

In part I of these lecture notes when deriving the predictive Kalman filter, it was shown that the conditional distribution of $x(t)$ given $I_1(t)$ is the same as the conditional distribution of $x(t)$ given $\hat{x}(t \mid t-1)$, where $\hat{x}(t \mid t-1)$ denotes the minimum variance estimate of $x(t)$ based on the information $I_1(t)$ (i.e. the optimal predicted value of $x(t)$ based on information up to time $t-1$). In addition, this conditional distribution is gaussian with mean value $\hat{x}(t \mid t-1)$ and covariance matrix $P_x(t)$ (see Part I of these lecture notes). Hence

$$\begin{gathered} E[x(t)^T Q_1 x(t) + u(t)^T Q_2 u(t) + V_{t+1}(I_1(t+1)) \mid I_1(t), u(t)] = \\ E[x(t)^T Q_1(t) + u(t)^T Q_2 u(t) + V_{t+1}(I_1(t+1)) \mid \hat{x}(t \mid t-1), u(t)]. \end{gathered}$$

Therefore we can define a new function

$$W_t(\hat{x}(t \mid t-1)) = V_t(I_1(t)), \quad t = N, \ldots, t_0.$$

Inserting this notation into (35),(36) and (37) while using the auxiliary result (29) gives

$$\min_{u(t_0),\ldots,u(N-1)} J_N = E[W_{t_0}(\hat{x}(t_0 \mid t_0 - 1))] \tag{38}$$

where $W_{t_0}(\cdot)$ is obtained recursively from the functional equation (for $t = N-1, \ldots, t_0$)

$$\begin{aligned} W_t(\hat{x}(t \mid t-1)) = \min\nolimits_{u(t)} \{ &\hat{x}(t \mid t-1)^T Q_1 \hat{x}(t \mid t-1) + \operatorname{tr} Q_1 P_x(t) + \\ &u(t)^T Q_2 u(t) + E[W_{t+1}(\hat{x}(t+1 \mid t)) \mid \hat{x}(t \mid t-1), u(t)] \} \end{aligned} \tag{39}$$

with the initial condition

$$W_N(\hat{x}(N \mid N-1)) = \hat{x}(N \mid N-1)^T Q_0 \hat{x}(N \mid N-1) + \operatorname{tr} Q_0 P_x(N). \tag{40}$$

Solution of the functional equation (39)

By analogy with (26), the functional equation (39) can be solved by showing that the solution has the form

$$W_t(\hat{x}(t \mid t-1)) = \hat{x}(t \mid t-1)^T S(t) \hat{x}(t \mid t-1) + s(t) \tag{41}$$

where $S(t)$ is a (symmetric) positive semidefinite (or positive definite) matrix and $s(t)$ is a scalar term. (We emphasize that these quantities are not assumed to be the same as in (28), although the notation is the same!)

By (40) we see that (41) holds for $t = N$ with $S(N) = Q_0$ and $s(N) = \operatorname{tr} Q_0 P_x(N)$. We shall show that if (41) holds for $t+1$, then it will also hold for t. By induction (41) then holds for $N, N-1, \ldots, t_0$. So assume that

$$W_{t+1}(\hat{x}(t+1 \mid t)) = \hat{x}(t+1 \mid t)^T S(t+1) \hat{x}(t+1 \mid t) + s(t+1)$$

In order to solve (39), we need to evaluate $E[W_{t+1}(\hat{x}(t+1 \mid t)) \mid \hat{x}(t \mid t-1), u(t)]$. By the Kalman filter equations, predictive case (see Part I of these lecture notes), it holds that

$$\hat{x}(t+1 \mid t) = A\hat{x}(t \mid t-1) + Bu(t) + K(t)(y(t) - C\hat{x}(t \mid t-1)), \tag{42}$$

where $K(t)$ is the Kalman filter gain given by

$$K(t) = AP_x(t)C^T(CP_x(t)C^T + R_2)^{-1}.$$

Here A, C and R_2 are quantities in the state space system (21), (22), (23) and $P_x(t)$ is the covariance matrix of the estimation error $x(t) - \hat{x}(t \mid t-1)$. Furthermore, the quantity

$$\tilde{y}(t) = y(t) - C\hat{x}(t \mid t-1)$$

($\{\tilde{y}(t)\}$ is the so-called innovation process) has a conditional distribution given $[I_1(t), u(t)]$, or $[\hat{x}(t \mid t-1), u(t)]$, which is gaussian with zero mean value and covariance matrix $CP_x(t)C^T + R_2$.

Hence it follows that $\hat{x}(t+1 \mid t)$ given $[\hat{x}(t \mid t-1), u(t)]$ is normally distributed with mean value

$$E[\hat{x}(t+1 \mid t) \mid \hat{x}(t \mid t-1), u(t)] = E[\hat{x}(t+1 \mid t) \mid I_1(t), u(t)] = A\hat{x}(t \mid t-1) + Bu(t)$$

and covariance matrix

$$E\Big[(\hat{x}(t+1 \mid t) - A\hat{x}(t \mid t-1) - Bu(t))\,(\hat{x}(t+1 \mid t) - A\hat{x}(t \mid t-1) - Bu(t))^T \mid$$
$$\hat{x}(t \mid t-1), u(t)\Big] = E\Big[K(t)\tilde{y}(t)\,(K(t)\tilde{y}(t))^T \mid \hat{x}(t \mid t-1), u(t)\Big] =$$
$$K(t)(CP_x(t)C^T + R_2)K(t)^T.$$

Thus by (29)

$$E[W_{t+1}(\hat{x}(t+1 \mid t)) \mid \hat{x}(t \mid t-1), u(t)] =$$
$$E[\hat{x}(t+1 \mid t)^T S(t+1) \hat{x}(t+1 \mid t) \mid \hat{x}(t \mid t-1), u(t)] + s(t+1) =$$
$$(A\hat{x}(t \mid t-1) + Bu(t))^T S(t+1) (A\hat{x}(t \mid t-1) + Bu(t)) +$$
$$\operatorname{tr}[S(t+1)K(t)(CP_x(t)C^T + R_2)K(t)^T] + s(t+1).$$

Inserting this into (39), we get that

$$W_t(\hat{x}(t \mid t-1)) = \min_{u(t)}\{\hat{x}(t \mid t-1)^T Q_1 \hat{x}(t \mid t-1) + \operatorname{tr} Q_1 P_x(t) +$$
$$u(t)^T Q_2 u(t) + (A\hat{x}(t \mid t-1) + Bu(t))^T S(t+1) (A\hat{x}(t \mid t-1) + Bu(t)) +$$
$$\operatorname{tr}[S(t+1)K(t)(CP_x(t)C^T + R_2)K(t)^T] + s(t+1)\} =$$
$$\min_{u(t)}\{u(t)^T (B^T S(t+1)B + Q_2)u(t) + u(t)^T B^T S(t+1)A\hat{x}(t \mid t-1) +$$
$$\hat{x}(t \mid t-1)^T A^T S(t+1)Bu(t)\} +$$
$$\hat{x}(t \mid t-1)^T (A^T S(t+1)A + Q_1)\hat{x}(t \mid t-1) + \operatorname{tr} Q_1 P_x(t) +$$
$$\operatorname{tr}[S(t+1)K(t)(CP_x(t)C^T + R_2)K(t)^T] + s(t+1),$$

where we have moved all terms that do not depend on $u(t)$ outside the minimization with respect to $u(t)$ operation. But here we have the same minimization problem as in (30) except that here $\hat{x}(t \mid t-1)$ replaces $x(t)$! Thus we can write down the solution directly from the earlier solution of (30). Thus a completion of squares argument gives that

$$W_t(\hat{x}(t \mid t-1)) =$$
$$\hat{x}(t \mid t-1)^T [A^T S(t+1)A - A^T S(t+1)B(B^T S(t+1)B + Q_2)^{-1} B^T S(t+1)A$$
$$+Q_1]\hat{x}(t \mid t-1) + \operatorname{tr} Q_1 P_x(t) + \operatorname{tr}[S(t+1)K(t)(CP_x(t)C^T + R_2)K(t)^T] + s(t+1),$$

and the minimum is attained for

$$(B^T S(t+1)B + Q_2)u(t) + B^T S(t+1)A\hat{x}(t \mid t-1) = 0$$

or

$$u(t) = -(B^T S(t+1)B + Q_2)^{-1} B^T S(t+1)A\hat{x}(t \mid t-1). \tag{43}$$

(In (43) $\hat{x}(t \mid t-1)$ replaces $x(t)$ in (31).) This means that $W_t(\hat{x}(t \mid t-1))$ is indeed given by (41) (and so our induction argument is complete). Furthermore, it is seen from the previously obtained expression for $W_t(\hat{x}(t \mid t-1))$ that $S(t)$ is given by the same equation (34) as in the complete state information case, and

$$s(t) = s(t+1) + \operatorname{tr}[Q_1 P_x(t)] + \operatorname{tr}[S(t+1)K(t)(CP_x(t)C^T + R_2)K(t)^T]$$

We summarize the solution to Problem 1 in the incomplete state information case (that corresponds to the predictive Kalman filtering situation) as follows.

Solution to Problem 1 with Incomplete State Information $I_1(t)$ — Summary

The admissible control strategies for $u(t)$ are allowed to be functions of the information state $I_1(t) = [y(t_0)^T, u(t_0)^T, \ldots, y(t-1)^T, u(t-1)^T]^T$ at time t.

The loss function (24) is minimized, among the admissible control strategies, by the control strategy

$$u(t) = -L(t)\hat{x}(t \mid t-1), \tag{44}$$

where $L(t)$ is given by (33) and the estimate $\hat{x}(t \mid t-1)$ is given by (42), see the treatment of the predictive Kalman filtering case in Part I of these lecture notes for the full details. If the initial state $x(t_0)$ has mean value m and covariance matrix R_0, the minimum of the loss function (24) is given by (putting $\hat{x}(t_0 \mid t_0 - 1) = m$)

$$\begin{gathered} \min_{u(t_0),\ldots,u(N-1)} J_N = E[W_{t_0}(\hat{x}(t \mid t-1))] = \\ E[\hat{x}(t_0 \mid t_0-1)^T S(t_0)\hat{x}(t_0 \mid t_0-1) + s(t_0)] = \\ m^T S(t_0) m + s(t_0) = m^T S(t_0) m + \textstyle\sum_{t=t_0}^{N-1} \mathrm{tr}[Q_1 P_x(t)] + \\ \textstyle\sum_{t=t_0}^{N-1} \mathrm{tr}[S(t+1)K(t)(CP_x(t)C^T + R_2)K(t)^T] + \mathrm{tr}[Q_0 P_x(N)], \end{gathered} \tag{45}$$

where we have used the previously obtained recursive expression for $s(t)$ and the initial value $s(N) = \mathrm{tr}[Q_0 P_x(N)]$. We recall from Part I of these lectures notes that $P_x(t)$, $t = t_0, \ldots, N$, is given by

$$P_x(t+1) = AP_x(t)A^T - AP_x(t)C^T(CP_x(t)C^T + R_2)^{-1}CP_x(t)A^T + R_1$$

with the initial value

$$P_x(t_0) = R_0.$$

Remark 3.5 Using the definition of the Kalman filter gain $K(t)$, the previous equation for $P_x(t)$ and (33)-(34), it can be shown that the minimum loss (45) can be written as

$$\begin{gathered} \min_{u(t_0),\ldots,u(N-1)} J_N = m^T S(t_0) m + \mathrm{tr}[S(t_0)R_0] + \\ \textstyle\sum_{t=t_0}^{N-1} \mathrm{tr}[R_1 S(t+1)] + \sum_{t=t_0}^{N-1} \mathrm{tr}[P_x(t)L(t)^T B^T S(t+1)A]. \end{gathered} \tag{46}$$

The first three terms give the minimum loss in the case of complete state information, cf. the summary to the solution of Problem 1 in the complete state information case. The fourth term is thus the additional loss due to the fact that information about the state is incomplete, so that only the information $I_1(t)$ is available when determining $u(t)$.

Remark 3.6 The previous result gives the optimal control strategy in the case when the disturbances of the state space system (21)-(22) are gaussian, i.e. normally distributed. (Recall that LQG means linear quadratic gaussian.) We observe that if it is not assumed that the disturbances are normally distributed (just that they are white noise with co-variances (23) and have zero mean), the results, in Part I of these lectures, for predictive

Kalman filtering still mean that (44) gives the optimal linear control law for the system. This follows as the predictive Kalman filter is the optimal linear filter for state estimation (i.e. for minimum variance estimation of $x(t)$ based on the information $I_1(t)$).

Dynamics of the Closed Loop System

The closed loop dynamics depends on both the dynamics of the Kalman filter for the estimate $\hat{x}(t \mid t-1)$ and the feedback from $\hat{x}(t \mid t-1)$. The closed loop system is described by the equations

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) + w(t) \\ y(t) &= Cx(t) + v(t) \\ u(t) &= -L(t)\hat{x}(t \mid t-1) \\ \hat{x}(t+1 \mid t) &= A\hat{x}(t \mid t-1) + Bu(t) + K(t)(y(t) - C\hat{x}(t \mid t-1)) \end{aligned}$$

Introduce the estimation error

$$\tilde{x}(t) = x(t) - \hat{x}(t \mid t-1).$$

The equations describing the closed loop system can then be written as

$$\begin{aligned} x(t+1) &= Ax(t) - BL(t)\hat{x}(t \mid t-1) + w(t) \\ &= (A - BL(t))x(t) + BL(t)\tilde{x}(t) + w(t) \\ \tilde{x}(t+1) &= A\tilde{x}(t) + w(t) - K(t)(y(t) - C\hat{x}(t \mid t-1)) \\ &= (A - K(t)C)\tilde{x}(t) + w(t) - K(t)v(t) \end{aligned}$$

In the stationary case the closed loop system can be written as, denoting $K = \lim_{t\to\infty} K(t)$ and $L = \lim_{N-t\to\infty} L(t)$,

$$\begin{pmatrix} x(t+1) \\ \tilde{x}(t+1) \end{pmatrix} = \begin{pmatrix} A - BL & BL \\ 0 & A - KC \end{pmatrix} \begin{pmatrix} x(t) \\ \tilde{x}(t) \end{pmatrix} + \begin{pmatrix} I \\ I \end{pmatrix} w(t) - \begin{pmatrix} 0 \\ K \end{pmatrix} v(t)$$

Note that the equation

$$\det \left[\begin{pmatrix} A - BL & BL \\ 0 & A - KC \end{pmatrix} - \lambda I \right] = \det(A - BL - \lambda I) \times \det(A - KC - \lambda I) = 0$$

determines the eigenvalues λ of the closed loop system matrix. Thus the closed loop eigenvalues consist of the eigenvalues of $A - BL$ and of $A - KC$, i.e. of the eigenvalues of the optimally controlled deterministic system and of the eigenvalues of the Kalman filter.

3.4 The Separation Principle

We shall discuss here briefly the celebrated separation principle of LQG control.

By Remark 3.4, the control law (32)-(33) is optimal for the deterministic linear quadratic (LQ) control problem. The optimal control strategy with incomplete state information consists of, according to (44):

1. An optimal state estimator to give the state estimate $\hat{x}$.

2. A linear feedback from the state estimate $\hat{x}$ using the optimal feedback gain matrix for the deterministic LQ control problem.

The linear quadratic gaussian (LQG) control problem thus has the property that estimation and control are separated. This is the celebrated **separation principle** of LQG control. This property is a consequence of the fact that the covariance matrix $P_x(t)$ of the estimation error does not depend on the observations, and hence, not on the inputs to the system.

3.5 Solution of the LQG Problem: Filtering Case

We have so far dealt with the incomplete state information case in the predictive case, i.e. in the case that the information available to compute $u(t)$ is given by $I_1(t)$, which contains $y(t-1)$ as the most recent available measurement. Here we shall consider the filtering case in which also $y(t)$ is available for computing $u(t)$, i.e. when the available information is given by $I_0(t)$.

We can write in vector form

$$I_0(t) = [y(t_0)^T, u(t_0)^T, \ldots, y(t-1)^T, u(t-1)^T, y(t)^T]^T.$$

We should emphasize that $u(t)$ does not belong to the information $I_0(t)$.

From the derivation of the solution to the LQG control problem (44), we see that it is possible to solve in an analogical manner the LQG control problem when an admissible control strategy is such that $u(t)$ is a function of $I_0(t)$. In fact, this observation can be also applied to the case when the available information for computing $u(t)$ is given by $I_k(t)$, where

$$I_k(t) = [y(t_0)^T, u(t_0)^T, \ldots, y(t-k)^T, u(t-k)^T, u(t-k+1)^T, \ldots, u(t-1)^T]$$

and $k = 1, 2, \ldots$ is a positive integer. (Then the most recent measurement available to compute $u(t)$ is $y(t-k)$.)

The loss function (24) is then minimized, among the admissible control strategies, by the control strategy

$$u(t) = -L(t)\hat{x}(t \mid t-k)$$

where $L(t)$ is given by (33)-(34). Note that here $k \geq 0$, i.e. also the case $k = 0$ is included. The optimal (minimum variance) predictive estimate of $x(t)$ given the information $I_k(t)$, for $k > 1$, is given analogously to the optimal predictive estimate $\hat{x}(t \mid t-1)$ derived in detail in Part I of these lecture notes, see also (42). The optimal filtering estimate $\hat{x}(t \mid t)$ ($k = 0$) was derived in Part I of these lecture notes (the Kalman filter in the filtering case).

We are mostly interested here in the case $k = 0$, and then the optimal control strategy is

$$u(t) = -L(t)\hat{x}(t \mid t), \tag{47}$$

where $L(t)$ is given by (33)-(34). As mentioned previously, $\hat{x}(t \mid t)$ is given by the Kalman filter equations in the filtering case as derived in Part I of these lecture notes.

Remark 3.7 If $y(t)$ is available for computing $u(t)$, then one should implement the control law (47), not (44), as (47) gives a smaller value for the control criterion (24). Furthermore, because in (44) $u(t)$ does not use $y(t)$, the control law (44) can give a larger value for the control criterion (24) than, say, a simple proportional (P-) controller $u(t) = Fy(t)$ for a well-chosen gain (matrix) F.

Stationary Solutions

Consider the stationary control law

$$u(t) = -L\hat{x}(t \mid t-1)$$

where L is given in Remark 3.2 and $\hat{x}(t \mid t-1)$ is given by the stationary form of the predictive Kalman filter, see Part I of these lecture notes. This stationary control law minimizes, among control laws $u(t)$ which are functions of $y(t-1)$, $y(t-2)$, …, $u(t-1)$, $u(t-2)$, …, the stationary loss function in Remark 3.2, i.e. the loss function

$$J = \lim_{N\to\infty} E\left[\frac{1}{N}\sum_{t=0}^{N-1}\left(x(t)^T Q_1 x(t) + u(t)^T Q_2 u(t)\right)\right]. \tag{48}$$

(That is, the average loss per step; it should be emphasized that in Remark 3.2 this loss function was minimized with respect to a different class of admissible control laws corresponding to complete state information.)

The minimum value of the stationary loss is by (46)

$$J_p = \min_{u(t)=f(y(t-1),y(t-2),\ldots,u(t-1),u(t-2),\ldots)} J = \mathrm{tr}[SR_1] + \mathrm{tr}[P_{x,p}L^T B^T SA],$$

where $P_{x,p} = P_x$ is the stationary covariance matrix of the estimation error in the predictive Kalman filter case, see Part I of these lecture notes. (Hence $P_{x,p}$ is the symmetric, positive semidefinite solution to the algebraic (stationary) Riccati equation for the covariance matrix of the estimation error of the predictive Kalman filter.)

Similarly consider the stationary control law

$$u(t) = -L\hat{x}(t \mid t),$$

where L is given in Remark 3.2 and $\hat{x}(t \mid t)$ is given by the stationary form of the filtering Kalman filter, see Part I of these lecture notes. This control law minimizes, among $u(t)$ which are functions of $y(t)$, $y(t-1)$, ..., $u(t-1)$, $u(t-2)$, ..., the loss function (48).

The minimum value of the stationary loss can be shown to be

$$J_f = \min_{u(t)=f(y(t),y(t-1),\ldots,u(t-1),u(t-2),\ldots)} J = \mathrm{tr}[SR_1] + \mathrm{tr}[P_{x,f}L^TB^TSA],$$

where $P_{x,f} = \lim_{t\to\infty} P_x(t \mid t)$ is the stationary covariance matrix of the estimation error $x(t) - \hat{x}(t \mid t)$ for the stationary form of the filtering Kalman filter.

Remark 3.8 Note that $L = (B^TSB + Q_2)^{-1}B^TSA$, so that

$$L^TB^TSA = A^TSB(B^TSB + Q_2)^{-1}B^TSA$$

is a symmetric positive semidefinite matrix. It holds that $\mathrm{tr}\, UV \geq 0$ for any (dimension compatible) U, V that are square, symmetric, and positive semidefinite matrices. By Part I of these lecture notes

$$P_{x,f} = P_{x,p} - P_{x,p}C^T(CP_{x,p}C^T + R_2)^{-1}CP_{x,p}$$

and so the difference

$$P_{x,p} - P_{x,f} = P_{x,p}C^T(CP_{x,p}C^T + R_2)^{-1}CP_{x,p} \geq 0$$

is a symmetric positive semidefinite matrix. Thus

$$\begin{aligned} J_p - J_f &= \mathrm{tr}[(P_{x,p} - P_{x,f})L^TB^TSA] = \\ &= \mathrm{tr}[P_{x,p}C^T(CP_{x,p}C^T + R_2)^{-1}CP_{x,p}A^TSB(B^TSB + Q_2)^{-1}B^TSA] \geq 0. \end{aligned}$$

That is, the stationary loss J_p is at least as large as J_f (as expected).

Exception to Separation in the Filtering Case

There is a case in which the optimality of the control law $u(t) = -L(t)\hat{x}(t \mid t)$ does not hold. This corresponds to the situation when the process noise and the measurement noise are correlated in the state space model. So we consider now the state space system

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) + w(t) \\ y(t) &= Cx(t) + v(t) \end{aligned}$$

with $R_{12} = E[w(t)v(t)^T] \neq 0$, but otherwise the state space system is as before.

We assume that the measurement signal and input signal information available to compute $u(t)$ is given by $I_0(t)$. As $R_{12} \neq 0$, it follows that there is in addition information available about the disturbance $w(t)$ to compute $u(t)$ via the measurement $y(t)$ (which is included in $I_0(t)$). Thus we can find an optimal estimate $\hat{w}(t \mid t)$ of the disturbance $w(t)$ which can now be nonzero. (In the case that $R_{12} = 0$ which was treated earlier, the minimum variance estimate of $w(t)$ given $I_0(t)$ is zero as then the conditional mean of $w(t)$ given $I_0(t)$ is equal to the unconditional mean of $w(t)$ and the latter is zero by assumption.)

As $w(t)$ affects $x(t+1)$, its estimate $\hat{w}(t \mid t)$ should also be fed back, and the optimal control strategy is not of the form $u(t) = -L(t)\hat{x}(t \mid t)$. That is, here the separation principle does not hold in its usual form! In the present case the information state $I_0(t)$ contains more information than $\hat{x}(t \mid t)$ about the future behavior of the state space system. (And therefore our earlier derivation of the optimal control strategy does not apply as such in this situation.)

This is exception is of particular interest, since it is common to start with a difference equation model of the ARX or ARMAX form treated in Part I of these lecture notes. This is true for example when the model is obtained via system identification. A popular state space realization of such difference equation models corresponds then to the case that $R_{12} \neq 0$, see Part I of these lecture notes for the details.

Example 2 Consider the time series model

$$y(t) - ay(t-1) = bu(t-1) + e(t) + ce(t-1),$$

where $\{e(t)\}$ is (possibly gaussian) white noise.

A natural state space representation of this model is (put $x(t) = y(t) - e(t)$)

$$\begin{aligned} x(t+1) &= ax(t) + bu(t) + (c+a)e(t) \\ y(t) &= x(t) + e(t) \end{aligned}$$

In this case the optimal control strategy can be found either by deriving the correct optimal control strategy which also feeds back $\hat{w}(t \mid t)$, or by transforming the system equations by expanding the state vector so as to make the process noise and measurement noise uncorrelated in the expanded model, and by applying the standard optimal control law to the expanded system! Let us proceed by the latter route.

Define the expanded state vector

$$x_e(t) = \begin{pmatrix} y(t) \\ e(t-1) \end{pmatrix}$$

This gives easily that

$$\begin{aligned} x_e(t+1) &= \begin{pmatrix} a & c \\ 0 & 0 \end{pmatrix} x_e(t) + \begin{pmatrix} b \\ 0 \end{pmatrix} u(t) + \begin{pmatrix} 1 \\ 1 \end{pmatrix} e(t) \\ y(t) &= \begin{pmatrix} 1 & 0 \end{pmatrix} x_e(t) \end{aligned}$$

Note that in this expanded model the measurement noise $v_e(t) = 0$ and so the covariance matrix $E[w_e(t)v_e(t)^T] = 0$, where the process noise of the expanded system is given by

$$w_e(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e(t)$$

Thus we can apply the standard form of the optimal control law to the expanded state system resulting in a control law of the form

$$u(t) = -L(t)\hat{x_e}(t \mid t)$$

(Note that one needs to add a fictitious measurement noise term, to the expanded model, with a small symmetric positive definite measurement noise matrix R_2' to guarantee the existence of a matrix inverse in the Riccati equation for the covariance matrix of the appropriate estimation error. One can put $R_2' = \delta \times I$, where $\delta > 0$ should be a very small number.)

3.6 Difference Equation Representation of Stationary Optimal Control Law

Consider the case when $t - t_0 \to \infty$, $N - t \to \infty$, and assume that both the Kalman filter gain $K(t)$ and the feedback gain matrix $L(t)$ approach their stationary values

$$K = \lim_{t-t_0 \to \infty} K(t), \quad L = \lim_{N-t \to \infty} L(t).$$

The control law (44) can be written as (with the stationary form of the predictive Kalman filter formula)

$$\begin{aligned} \hat{x}(t+1 \mid t) &= A\hat{x}(t \mid t-1) + Bu(t) + K(y(t) - C\hat{x}(t \mid t-1)) \\ &= (A - BL - KC)\hat{x}(t \mid t-1) + Ky(t) \\ u(t) &= -L\hat{x}(t \mid t-1) \end{aligned}$$

But these equations define a state space representation of a linear system with input y and output u, and hence these equations can be written via the transfer function (from y to u) in the difference equation form

$$u(t) + H_1 u(t-1) + \ldots + H_j u(t-j) = G_1 y(t-1) + \ldots + G_k y(t-k).$$

(Here j and k can be chosen to be smaller all equal to n, the dimension of the vector $\hat{x}$.) Note that $y(t)$ is not fed back to $u(t)$ ($y(t)$ is not available for $u(t)$).

Analogously, the control law (47) can be written as (with the stationary form of the filtering Kalman filter formula)

$$\begin{aligned} \hat{x}(t+1 \mid t+1) &= A\hat{x}(t \mid t) + Bu(t) + \bar{K}[y(t+1) - C(A\hat{x}(t \mid t) + Bu(t))] \\ &= (A - BL - \bar{K}CA + \bar{K}CBL)\hat{x}(t \mid t) + \bar{K}y(t+1) \\ u(t) &= -L\hat{x}(t \mid t) \end{aligned}$$

This can be written in difference equation form as

$$u(t) + \bar{H}_1 u(t-1) + \ldots + \bar{H}_j u(t-j) = \bar{G}_0 y(t) + \ldots + \bar{G}_k y(t-k).$$

Note that $y(t)$ IS here fed back to $u(t)$!

3.7 Stationary Solutions of the Riccati Equation

We noted earlier that the Riccati equations of (predictive) Kalman filtering and linear quadratic control have essentially the same mathematical structure. Thus we consider the stationary form of the Riccati equation (34) only.

The stationary form of this Riccati equation is

$$\begin{aligned} S &= A^T SA - A^T SB(B^T SB + Q_2)^{-1} B^T SA + Q_1 \\ &= (A - BL)^T S(A - BL) + L^T Q_2 L + Q_1, \end{aligned}$$

where $L = (B^T SB + Q_2)^{-1} B^T SA$ is the feedback gain matrix. The closed loop system becomes with $u(t) = -Lx(t)$

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) + w(t) \\ &= (A - BL)x(t) + w(t) \end{aligned}$$

We are only interested in such solutions of the stationary (or algebraic) Riccati equation, which make the closed loop stable, i.e. the matrix $A - BL$ must have all its eigenvalues strictly inside the unit circle. We must note that this restriction is important. There are cases in which the algebraic Riccati equation can have a positive semidefinite solution giving a feedback gain matrix L such that the closed loop system is not stable. (An example is provided by the minimum variance control law for nonminimum phase systems that gives the global minimum for the output variance: the input part of the closed loop system is then unstable resulting in an input that grows with time and hence then the control law $u(t) = -Lx(t)$ should not be implemented.)

That is, there can be important situations in which the algebraic Riccati equation, a nonlinear equation in S, has several solutions, even several positive semidefinite solutions. It is important to check that one uses the correct (stabilizing) solution of the algebraic Riccati equation. Use professionally made software to solve algebraic Riccati equations!

4 Concluding Remarks

This course has dealt with minimum variance state estimation, that is Kalman filtering, and linear quadratic gaussian (LQG) control. Fairly detailed derivations of the Kalman filter and the LQG control law have been given. We have also given the necessary background material on minimum variance estimation and dynamic programming (also stochastic dynamic programming). This should make it possible for the student to derive optimal state estimators and optimal control laws in other related situations not covered in this course.

Kalman filtering and LQG control are two of the most elegant and versatile methods in the control and systems area. With the rapid increase in the complexity of estimation and control applications in networked systems, it is expected that the range of further developments and applications of Kalman filtering and LQG control continues to grow in the future.